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Conformal current algebra in two dimensions

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Abstract: We construct a non-chiral current algebra in two dimensions consistent with conformal invariance. We show that the conformal current algebra is realized in non-linear sigma-models on supergroup manifolds with vanishing Killing form, with or without a Wess-Zumino term. The current algebra is computed using two distinct methods. First we exploit special algebraic properties of supergroups to compute the exact two- and three-point functions of the currents and from them we infer the current algebra. The algebra is also calculated by using conformal perturbation theory about the Wess-Zumino-Witten point and resumming the perturbation series. We also prove that these models realize a non-chiral Kac-Moody algebra and construct an infinite set of commuting operators that is closed under the action of the Kac-Moody generators. The supergroup models that we consider include models with applications to statistical mechanics, condensed matter and string theory. In particular, our results may help to systematically solve and clarify the quantum integrability of $PSU(n|n)$ models and their cosets, which appear prominently in string worldsheet models on anti-deSitter spaces.

Keywords: Conformal Field Models in String Theory, Conformal and W Symmetry, AdS-CFT Correspondence

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Contents

1 Introduction

Two-dimensional sigma-models on supergroups have applications to a wide range of topics such as the integer quantum hall effect, quenched disorder systems, polymers, string theory, as well as other domains in physics (see e.g. [1–5]). The principal chiral model is perturbatively conformal on various supergroups [6, 7] with or without the addition of a Wess-Zumino term, and at least to two loop order on their cosets with respect to a maximal regular subalgebra [7]. Sigma models on graded supercosets are also believed to be conformal [8]. Thus, these models have an infinite dimensional symmetry algebra that should tie in with their supergroup symmetry. An extended non-linear symmetry algebra was identified in [6] but the representation theory of the algebra seems difficult to establish. Steps towards solving these models were made using various techniques [9–11]. In this paper, we exhibit a conformal current algebra in these models. The algebra of currents is non-chiral and implies conformal symmetry, hence the name.

The models under discussion enter as the key building blocks in worldsheet sigma-models on supersymmetric AdS backgrounds in string theory. The supergroup $PSU(1,1|2)$ principal chiral model corresponds to a supersymmetric $AdS_3 \times S^3$ background with Ramond-Ramond flux [6, 12]. Since a theory of quantum gravity on asymptotically AdS_3 space-times, supplemented with appropriate boundary conditions, exhibits an infinite dimensional conformal symmetry algebra [13], we should be able to construct those generators from the worldsheet theory. Indeed, for AdS_3 string theory with only Neveu-Schwarz-Neveu-Schwarz flux, it has been shown how to construct the space-time Virasoro algebra in terms of the worldsheet current algebra [14–16]. To perform a similar construction in Ramond-Ramond backgrounds, one needs to understand the worldsheet current algebra for two-dimensional models with supergroup targets.

A second application within this context is the extension of our analyis to supercoset manifolds, which includes the $AdS_5 \times S^5$ background of string theory. The worldsheet current algebra is tied in with the integrability of the worldsheet theory [17]. Our work may help in systematically exploiting the integrability of the worldsheet model at the quantum level, with applications to the solution of the spectrum of planar four-dimensional gauge theories (see e.g. [18]) via the AdS/CFT correspondence.

The plan of the paper is as follows. In section 2 we kick off with a general analysis of two-dimensional current algebra operator product expansions that are consistent with locality, Lorentz invariance and parity-time reversal. We exhibit a particular current algebra that is non-chiral and consistent with conformal invariance. We then move on to exhibit a realization of the algebra in a conformally invariant model on a supergroup manifold.

We calculate perturbatively the current two- and three-point functions of these models in section 3. From these correlators we infer the operator product expansions of the currents, which are shown to fall into the conformal current algebra class discussed in section 2. In section 4 we analyze deformed Wess-Zumino-Witten models on supergroups using conformal perturbation theory. We compute the operator product expansions of the currents to all orders and resum the series, thereby demonstrating that the resulting algebra matches the results obtained using purely algebraic properties of the supergroups. We analyze the current algebra on the cylinder in terms of a Fourier decomposition in section 5 and show the existence of a Kac-Moody subalgebra and an infinite set of commuting operators that transform amongst each other under the Kac-Moody subalgebra. In section 6 we discuss some applications of the conformal current algebra that we have found and possible future directions of work.

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2 Current algebras in two dimensions

Before we compute the current algebra operator product expansion [19] for supergroup sigma-models, it is interesting to analyze the generic operator product expansions (OPEs) involving the currents for a two-dimensional model which is local, Lorentz invariant and which respects parity-time reversal.

Previously, the generic two-dimensional current algebra was analyzed in [20] where it was applied to an asymptotically free sigma-model — the $O(n)$ sigma-model. Parity symmetry was assumed to be valid in the analysis. Later, a similar analysis was performed [21, 22] for massive models with Wess-Zumino-Witten ultraviolet fixed points. In both cases, the study was applied to argue for the integrability of the two-dimensional sigma-model in the quantum theory.

The models that we will study have the distinctive feature that they are conformal. Moreover, the Wess-Zumino term breaks parity invariance. Therefore, we start by analyzing the generic current operator product expansions consistent with locality, Lorentz invariance and PT-invariance only. In the following, we generalize the methodology of [20].

2.1 Locality, Lorentz invariance and PT-invariance

In the absence of parity invariance, the vector representation of the two-dimensional Lorentz group splits into two one-dimensional irreducible representations. A current j_μ can therefore be split into two irreducible representations j_+ and j_- of the two-dimensional Lorentz group.[1] We write the OPEs in terms of these irreducible components, leading to three independent OPEs, between the pairs of current components (j_+, j_+), (j_+, j_-), and (j_-, j_-).

We first analyze the OPE between the components j_+ and j_+. We take the currents to be in the adjoint representation of a symmetry group G of the model: $j_+ = j_+^a t_a$ where t_a spans the Lie algebra of G. We write down the generic OPE in terms of Lorentz invariant coefficient functions. Moreover, we assume that the only low-dimensional operators that appear in the operator product expansion are the identity operator, the currents and their derivatives. We also assume that the currents have conformal dimension one. Thus the previous list of allowed operators in the OPE should account for all the terms up to regular ones. The $j_+ j_+$ OPE is then given by:

$$\begin{aligned} j_+^a(x) j_+^b(0) \sim\ & \alpha^{ab}(x^-)^2 d_1(x^2) + \beta^{ab}{}_c (x^-)^2 \Big[d_2(x^2) x^+ j_+^c(0) + d_3(x^2) x^- j_-^c(0) \\ & + e_1(x^2) x^+ x^- \partial_+ j_-^c(0) + e_2(x^2) x^+ x^- \partial_- j_+^c(0) \\ & + e_3(x^2)(x^+)^2 \partial_+ j_+^c(0) + e_4(x^2)(x^-)^2 \partial_- j_-^c(0) \Big] + \ldots \end{aligned} \tag{2.1}$$

where the functions d_i, e_i are functions of the Lorentz invariant $x^2 = x^+ x^-$. The tensor α^{ab} is an invariant two-tensor in the product of adjoint representations, and $\beta^{ab}{}_c$ represents an adjoint representation in the product of two adjoints. We will assume that α^{ab} corresponds to a non-degenerate bi-invariant metric κ^{ab} and that the tensor $\beta^{ab}{}_c$ is equal to the structure

[1] We work in Lorentzian signature in this section. We define $x^\pm = x \pm t$ and $ds^2 = -dt^2 + dx^2 = dx^+ dx^-$. We denote $x^2 = x^+ x^-$.

constants $f^{ab}{}_c$ of the Lie algebra of G. We take the structure constants to be anti-symmetric in its indices.[2] We use the fact that we can interchange operators[3] to determine that the above OPE should be equal to:

$$\begin{aligned} j_+^b(0)j_+^a(x) \sim\ & \kappa^{ba}(x^-)^2 d_1 - f^{ba}{}_c(x^-)^2\Big(d_2 x^+ j_+^c(x) + d_3 x^- j_-^c(x) \\ & -e_1 x^+x^-\partial_+ j_-^c(x) - e_2 x^+x^-\partial_- j_+^c(x) - e_3(x^+)^2\partial_+ j_+^c(x) - e_4(x^-)^2\partial_- j_-^c(x)\Big) \\ \sim\ & \kappa^{ba}x^-x^- d_1 - f^{ba}{}_c x^-x^-\Big(d_2 x^+ j_+^c(0) + d_3 x^- j_-^c(0) \\ & +(d_3-e_1)x^+x^-\partial_+ j_-^c(x) + (d_2-e_2)x^+x^-\partial_- j_+^c(x) \\ & +(d_2-e_3)x^+x^+\partial_+ j_+^c(x) + (d_3-e_4)x^-x^-\partial_- j_-^c(x)\Big) \end{aligned} \tag{2.2}$$

from which we derive the equations:

$$\begin{aligned} e_1 &= d_3/2 \qquad & e_2 &= d_2/2 \\ e_3 &= d_2/2 \qquad & e_4 &= d_3/2, \end{aligned} \tag{2.3}$$

which gives rise to the simplified operator product expansion:

$$\begin{aligned} j_+^a(x)j_+^b(0) \sim & \kappa^{ab}(x^-)^2 d_1 + f^{ab}{}_c(x^-)^2\left[d_2 x^+ j_+^c(0) + d_3 x^- j_-^c(0) + \frac{d_3}{2}x^+x^-\partial_+ j_-^c(0)\right. \\ & \left. +\frac{d_2}{2}x^+x^-\partial_- j_+^c(0) + \frac{d_2}{2}(x^+)^2\partial_+ j_+^c(0) + \frac{d_3}{2}(x^-)^2\partial_- j_-^c(0)\right] + \dots \end{aligned} \tag{2.4}$$

The OPE has one free coefficient d_1 (function of x^2) at leading order, and two at subleading order. We similarly obtain:

$$\begin{aligned} j_-^a(x)j_-^b(0) \sim & \kappa^{ab}(x^+)^2 d_4 + f^{ab}{}_c(x^+)^2\left[d_5 x^- j_-^c(0) + d_6 x^+ j_+^c(0) + \frac{d_5}{2}x^+x^-\partial_+ j_-^c(0)\right. \\ & \left. +\frac{d_5}{2}(x^-)^2\partial_- j_-^c(0) + \frac{d_6}{2}x^+x^-\partial_- j_+^c(0) + \frac{d_6}{2}(x^+)^2\partial_+ j_+^c(0)\right] + \dots \end{aligned} \tag{2.5}$$

For the OPE between j_+ and j_- we don't get as many constraints. We find 7 more free functions:

$$\begin{aligned} j_+^a(x)j_-^b(0) \sim \kappa^{ab}d_7 + f^{ab}{}_c\Big[& d_8 x^+ j_+^c(0) + d_9 x^- j_-^c(0) + d_{13}x^+x^-\partial_+ j_-^c(0) + d_{12}x^+x^-\partial_- j_+^c(0) \\ & +d_{10}(x^+)^2\partial_+ j_+^c(0) + d_{11}(x^-)^2\partial_- j_-^c(0)\Big] + \dots \end{aligned} \tag{2.6}$$

We have a total of 13 free coefficient functions.

[2] More precisely, we take the structure constants to be graded anti-symmetric when G is a supergroup. Although the grading is crucial, it will not affect our formulas, except for a plethora of minus signs when interchanging operators — these will not influence our final results much. We maintain consistency with the grading throughout section 2, but not necessarily through the rest of the paper.

[3] Up to a minus sign for fermionic operators. We see, for example, that the interchange of fermionic operators will cancel a minus sign from the grading of the superalgebra when G is a supergroup.

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2.2 Current conservation

We now impose consistency of the operator product expansions of the currents with current conservation. We choose the relative normalization of the two components of the currents such that the equation of current conservation reads:[4]

$$\partial^\mu j^a_\mu = \partial_- j^a_+ + \partial_+ j^a_- = 0. \tag{2.7}$$

Current conservation implies that one of the coefficients in the $j_+ j_-$ OPE (namely $d_{12}+d_{13}$) becomes redundant. We check that the OPEs of the current conservation equation (2.7) with the currents j_+ and j_- vanish. That leads to the set of equations:[5]

$$\begin{aligned}
&d_1 + \frac{x^2}{2}d_1' + \frac{1}{2}d_7' = 0\,, \qquad && d_2 + \frac{x^2}{2}d_2' + \frac{1}{2}d_8' + \frac{d_8}{2x^2} = 0\,,\\
&d_4 + \frac{x^2}{2}d_4' + \frac{1}{2}d_7' = 0\,, && \frac{3}{2}d_6 + \frac{x^2}{2}d_6' + \frac{1}{2}d_8' = 0\,,\\
&\frac{3}{2}d_3 + \frac{x^2}{2}d_3' + \frac{1}{2}d_9' = 0\,, && 2d_3 + \frac{x^2}{2}d_3' + (d_9' - d_{11}') = 0\\
&2d_6 + \frac{x^2}{2}d_6' + d_{10}' = 0\,, && d_5 + \frac{x^2}{2}d_5' + d_{11}' + \frac{2}{x^2}d_{11} = 0\,,\\
&d_5 + \frac{x^2}{2}d_5' + \frac{1}{2}d_9' + \frac{1}{2x^2}d_9 = 0\,, &&\\
&d_2 + \frac{x^2}{2}d_2' + (d_8' - d_{10}') + \frac{2}{x^2}(d_8 - d_{10}) = 0 &&\\
&\frac{3}{2}(d_6 - d_5) + \frac{x^2}{2}(d_6' - d_5') + (d_{12}' - d_{13}') + \frac{1}{x^2}(d_{12} - d_{13}) = 0\,, &&\\
&\frac{3}{2}(d_2 - d_3) + \frac{x^2}{2}(d_2' - d_3') + (d_8' - d_{12}' - (d_9' - d_{13}')) + \frac{1}{x^2}(d_8 - d_{12} - (d_9 - d_{13})) = 0\,. &&
\end{aligned} \tag{2.8}$$

We get a system of twelve first-order differential equations for twelve functions. We will not try to solve them in full generality. Instead, we assume that the leading and subleading singularities in the OPEs are powerlike. This leads to the following ansatz:

$$\begin{aligned}
&d_1(x^2) = c_1/x^4, \quad d_2(x^2) = c_2/x^4, \quad d_3(x^2) = f_1/x^4\,,\\
&d_4(x^2) = c_3/x^4, \quad d_5(x^2) = c_4/x^4, \quad d_6(x^2) = f_7/x^4,\\
&d_7(x^2) = f_2/x^2, \quad d_8(x^2) = f_3/x^2, \quad d_9(x^2) = f_4/x^2\,.
\end{aligned} \tag{2.9}$$

The coefficient c_i's and f_i's are now constant coefficients. Plugging this ansatz into the equations (2.8), we get the following relations between the coefficients:

$$f_2 = 0, \qquad f_4 = f_1, \qquad f_3 = f_7 = c_4 - c_2 + f_1\,. \tag{2.10}$$

[4]We expect current conservation to only hold up to delta-function contact terms. We therefore do not keep track of contact terms in the following.

[5]There is no equation corresponding to the operator $\partial_\mu j^\mu$ since the coefficient multiplies zero.

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The remaining equations in (2.8) then allow to solve for the subsubleading coefficient functions in the OPEs:

$$\begin{aligned} d_{10}(x^2) &= \frac{f_7}{x^2}, \qquad d_{11}(x^2) = 0, \\ (d'_{12} - d'_{13}) + \frac{1}{x^2}(d_{12} - d_{13}) &= \frac{c_2 - f_1}{2x^4}. \end{aligned} \tag{2.11}$$

So it only remains to solve for $d_{12}(x^2) - d_{13}(x^2)$ in terms of the c_i and f_1. Denoting $c_2 - f_1 = g$, the general solution to the differential equation (2.11) reads:

$$d_{12}(x^2) - d_{13}(x^2) = \frac{2c_5}{x^2} + \frac{g}{2x^2} \log \mu^2 x^2 \tag{2.12}$$

where c_5 is a constant coefficient and μ is an arbitrary mass scale. We note that when the coefficient g is non-zero, we can absorb the coefficient c_5 in a redefinition of the mass scale μ. So the OPEs are given in terms of five dimensionless coefficients. They read:

$$\begin{aligned} j_+^a(x) j_+^b(0) &\sim \frac{\kappa^{ab} c_1}{(x^+)^2} + f^{ab}{}_c \left[\frac{c_2}{x^+} j_+^c(0) + \frac{(c_2 - g) x^-}{(x^+)^2} j_-^c(0) - \frac{g}{4} \frac{x^-}{x^+} \left(\partial_+ j_-^c(0) - \partial_- j_+^c(0) \right) \right. \\ &\qquad \left. + \frac{c_2}{2} \partial_+ j_+^c(0) + \frac{c_2 - g}{2} \frac{(x^-)^2}{(x^+)^2} \partial_- j_-^c(0) \right] + \ldots \\ j_-^a(x) j_-^b(0) &\sim \frac{\kappa^{ab} c_3}{(x^-)^2} + f^{ab}{}_c \left[\frac{c_4}{x^-} j_-^c(0) + \frac{(c_4 - g)\, x^+}{(x^-)^2} j_+^c(0) + \frac{g}{4} \frac{x^+}{x^-} \left(\partial_+ j_-^c(0) - \partial_- j_+^c(0) \right) \right. \\ &\qquad \left. + \frac{c_4}{2} \partial_- j_-^c(0) + \frac{c_4 - g}{2} \frac{(x^+)^2}{(x^-)^2} \partial_+ j_+^c(0) \right] + \ldots \\ j_+^a(x) j_-^b(0) &\sim f^{ab}{}_c \left[\frac{c_4 - g}{x^-} j_+^c(0) + \frac{(c_2 - g)}{x^+} j_-^c(0) + \frac{(c_4 - g) x^+}{x^-} \partial_+ j_+^c(0) \right. \\ &\qquad \left. - \left(c_5 + \frac{g}{4} \log \mu^2 x^2 \right) \left(\partial_+ j_-^c(0) - \partial_- j_+^c(0) \right) \right] + \ldots \end{aligned} \tag{2.13}$$

It would be interesting to search for more general solutions to the set of differential equations.

2.3 The Maurer-Cartan equation

In this subsection we show that under certain circumstances, we can obtain further constraints on the current algebra. Consider a field g taking values in a Lie group. The one-form dgg^{-1} satisfies the Maurer-Cartan equation

$$d(dgg^{-1}) = dgg^{-1} \wedge dgg^{-1}. \tag{2.14}$$

We will get further constraints if we suppose that the components of the current are related to the field g in the following way:

$$j_+ = c_+ \partial_+ g g^{-1}, \quad j_- = c_- \partial_- g g^{-1} \tag{2.15}$$

where c_+ and c_- are constant coefficients. The generators of the Lie (super-)algebra satisfy: $[t_a, t_b] = if^c{}_{ab}t_c$. Then the Maurer-Cartan equation takes the form

$$c_-\partial_- j_+^a - c_+\partial_+ j_-^a - if^a{}_{bc} j_+^c j_-^b = 0. \tag{2.16}$$

We want to ensure that this equation is also valid in the quantum theory. However the operator defined as the product of two currents needs to be regularized. For this reason the validity of the Maurer-Cartan equation in the quantum theory requires more discussion.

Normal ordering. In the quantum theory, we introduce a normal ordering for composite operators based on a point-splitting procedure. The normal ordered product $: O_1 O_2 : (y)$ of two operators O_1 and O_2 evaluated at the point y is defined to be the product of the operator O_1 at the point x with the operator O_2 at the point y, in the limit as x approaches y. The regularization amounts to dropping the terms that are singular in this limit. For this procedure to be well-defined, it is important that the resulting operator is evaluated at the point y. We will denote this procedure by

$$: O_1 O_2 : (y) = \lim_{:x \to y:} O_1(x) O_2(y)\,. \tag{2.17}$$

We note that the operators within the normal ordered product $: O_1 O_2 :$ do not commute.[6] We will later confirm that a natural choice for the normal ordered Maurer-Cartan equation in this scheme is:[7]

$$c_-\partial_- j_+^a - c_+\partial_+ j_-^a - \frac{i}{2} f^a{}_{bc}\left(: j_+^c j_-^b : + (-)^{bc} : j_-^b j_+^c : \right) = 0. \tag{2.18}$$

Additional constraints from the Maurer-Cartan equation. As for the current conservation equation, we ask for the OPE between the quantum Maurer-Cartan equation (2.18) and the current to vanish. The first non-trivial constraint is obtained for the subleading terms. This leads to a relation between the coefficient of the current algebra c_1, c_2 and g, and the coefficients c_+ and c_-:

$$(c_+ + c_-)(c_2 - g) + ic_1 = 0\,. \tag{2.19}$$

We similarly find a constraint linking c_3, c_4 and g to c_+ and c_-:

$$(c_+ + c_-)(c_4 - g) + ic_3 = 0\,. \tag{2.20}$$

When we consider concrete models realizing the current algebra, the coefficients c_+ and c_- can be derived from the Lagrangian. The Maurer-Cartan equation then reduces the number of free constant coefficients from five to three.

[6] See e.g. [23] for a discussion of this fact in the context of chiral conformal field theories.
[7] We use the notation $(-)^a = +1$ if a is a bosonic index, and -1 if a is a fermionic index.

2.4 The Euclidean current algebra

For future purposes, we wish to translate the result we obtained for the current operator algebra into euclidean signature. We perform the Wick rotation $t \to -i\tau$, and define the complex coordinates $z = x - i\tau$, $\overline{z} = x + i\tau$. The current algebra OPEs become

$$\begin{aligned}
j_z^a(z)j_z^b(0) \sim & \kappa^{ab}\frac{c_1}{z^2} + f^{ab}{}_c\left[\frac{c_2}{z}j_z^c(0) + (c_2 - g)\frac{\overline{z}}{z^2}j_{\overline{z}}^c(0)\right. \\
& \left. -\frac{g}{4}\frac{\overline{z}}{z}(\partial_z j_{\overline{z}}^c(0) - \partial_{\overline{z}} j_z^c(0)) + \frac{c_2}{2}\partial_z j_z^c(0) + \frac{c_2 - g}{2}\frac{\overline{z}^2}{z^2}\partial_{\overline{z}} j_{\overline{z}}^c(0)\right] + \dots \\
j_{\overline{z}}^a(z)j_{\overline{z}}^b(0) \sim & \kappa^{ab}c_3\frac{1}{\overline{z}^2} + f^{ab}{}_c\left[\frac{c_4}{\overline{z}}j_{\overline{z}}^c(0) + \frac{(c_4 - g)z}{\overline{z}^2}j_z^c(0)\right. \\
& \left. +\frac{g}{4}\frac{z}{\overline{z}}(\partial_z j_{\overline{z}}^c(0) - \partial_{\overline{z}} j_z^c(0)) + \frac{c_4}{2}\partial_{\overline{z}} j_{\overline{z}}^c(0) + \frac{(c_4 - g)}{2}\frac{z^2}{\overline{z}^2}\partial_z j_z^c(0)\right] + \dots \\
j_z^a(z)j_{\overline{z}}^b(0) \sim & f^{ab}{}_c\left[\frac{(c_4 - g)}{\overline{z}}j_z^c(0) + \frac{(c_2 - g)}{z}j_{\overline{z}}^c(0) + \frac{(c_4 - g)z}{\overline{z}}\partial_z j_z^c(0)\right. \\
& \left. -(c_5 + \frac{g}{4}\log\mu^2|z|^2)(\partial_z j_{\overline{z}}^c(0) - \partial_{\overline{z}} j_z^c(0))\right] + \dots
\end{aligned} \tag{2.21}$$

Later on, we will compare the current algebra operator product expansions in equations (2.21) to those of a supergroup non-linear sigma-model with Wess-Zumino term. We will find specific expressions for the coefficients c_i and g in terms of the parameters in the Lagrangian.

2.5 Conformal current algebra

It turns out that the above current algebra can become the building unit for a conformal algebra when the Killing form of the (super-)group vanishes.[8] In that special case the current algebra is promoted to a conformal current algebra, namely, the Sugawara stress-energy tensor built from the currents satisfies the canonical conformal operator product expansion. The terms that in other circumstances spoil conformality are eliminated through the fact that the Killing form is zero. The holomorphic component of the stress-energy tensor is

$$T(w) = \frac{1}{2c_1} : j_{bz} j_z^b : (w), \tag{2.22}$$

as we will demonstrate.[9]

[8] For simple super Lie algebras the vanishing of the Killing form is equivalent to the vanishing of the dual Coxeter number.

[9] We defined $j_a = j^b \kappa_{ba}$ and $\kappa^{ac}\kappa_{cb} = \delta^a{}_b$. In the following we also use the convention $f_{abc} = f_{ab}{}^d \kappa_{dc}$.

The current as a conformal primary. First we compute the OPE between the current j^a_z and its bilinear combination $: j_{bz} j^b_z :$, using a point-splitting procedure:

$$
\begin{aligned}
j^a_z(z) : j_{bz} j^b_z : (w) &= \lim_{:x \to w:} j^a_z(z) \left[j_{bz}(x) j^b_z(w) \right] \\
&= \lim_{:x \to w:} \Bigg[\left(\frac{c_1 \delta^a_b}{(z-x)^2} + f^a{}_{bc} \left(\frac{c_2}{z-x} j^c_z(x) + \frac{(c_2 - g)(\overline{z} - \overline{x})}{(z-x)^2} j^c_{\overline{z}}(x) \right) \right) j^b_z(w) \\
&\qquad + j^d_z(x) (-1)^{ab} \kappa_{db} \Bigg(\frac{c_1 \kappa^{ab}}{(z-w)^2} + f^{ab}{}_c \\
&\qquad \times \left(\frac{c_2}{z-w} j^c_z(w) + \frac{(c_2 - g)(\overline{z} - \overline{w})}{(z-w)^2} j^c_{\overline{z}}(w) \right) \Bigg) \Bigg] .
\end{aligned}
\tag{2.23}
$$

At this point we have to perform the OPE between the operators evaluated at the point x and the operators evaluated at the point w. Then we take the limit where x goes to w and discard the singular terms. We notice already that only the regular terms in the OPEs of the second line will contribute to the final result. We get

$$
\begin{aligned}
j^a_z(z) : j_{bz} j^b_z : (w) = \lim_{:x \to w:} \Bigg[& c_1 \frac{j^a_z(w)}{(z-x)^2} + \frac{c_2 f^a{}_{bc}}{z-x} \Bigg(\frac{c_1 \kappa^{cb}}{(x-w)^2} \\
& + f^{cb}{}_d \left(\frac{c_2 j^d_z(w)}{x-w} + (c_2 - g) \frac{\overline{x} - \overline{w}}{(x-w)^2} j^d_{\overline{z}}(w) \right) + : j^c_z j^b_z : (w) \Bigg) \\
& + \frac{(c_2 - g) f^a{}_{bc} (\overline{z} - \overline{x})}{(z-x)^2} \Bigg(f^{cb}{}_d \Bigg(\frac{(c_4 - g) j^d_z(w)}{\overline{x} - \overline{w}} + \frac{(c_2 - g) j^d_{\overline{z}}(x)}{x - w} \\
& - \left(c_5 + \frac{g}{4} \log \mu^2 |x-w|^2 \right) \left(\partial_z j^d_{\overline{z}}(w) - \partial_{\overline{z}} j^d_z(w) \right) \Bigg) + : j^c_{\overline{z}} j^b_z : (w) \Bigg) \Bigg] \\
& + c_1 \frac{j^a_z(w)}{(z-w)^2} + (-1)^{bc} f^a{}_{bc} \frac{c_2}{z-w} : j^b_z j^c_z : (w) \\
& + (-1)^{bc} f^a{}_{bc} \frac{(c_2 - g)(\overline{z} - \overline{w})}{(z-w)^2} : j^b_z j^c_{\overline{z}} : (w) .
\end{aligned}
\tag{2.24}
$$

Among the remaining terms, many cancel: every contraction of the invariant metric with a structure constant gives zero by symmetry, and the double contractions of two structure constants are proportional to the Killing form and thus also vanish. We are left with

$$
\begin{aligned}
j^a_z(z) : j_{bz} j^b_z : (w) = \; & 2c_1 \frac{j^a_z(w)}{(z-w)^2} + c_2 \frac{f^a{}_{bc}}{z-w} \left((-1)^{bc} : j^b_z j^c_z : (w) + : j^c_z j^b_z : (w) \right) \\
& + (c_2 - g) f^a{}_{bc} \frac{\overline{z} - \overline{w}}{(z-w)^2} \left((-1)^{bc} : j^b_z j^c_{\overline{z}} : (w) + : j^c_{\overline{z}} j^b_z : (w) \right) .
\end{aligned}
\tag{2.25}
$$

The second term vanishes because of the anti-(super)symmetry of the structure constants. We can simplify the third term using the Maurer-Cartan identity:

$$
j^a_z(z) : j_{bz} j^b_z : (w) = 2c_1 \frac{j^a_z(w)}{(z-w)^2} + 2i(c_2 - g) \frac{\overline{z} - \overline{w}}{(z-w)^2} \left(c_- \overline{\partial} j^a_z(w) - c_+ \partial j^a_{\overline{z}}(w) \right) . \tag{2.26}
$$

By current conservation this can be rewritten as:

$$j_z^a(z) : j_{bz} j_z^b : (w) = 2c_1 \frac{j_z^a(w)}{(z-w)^2} + 2i(c_2 - g) \frac{\overline{z} - \overline{w}}{(z-w)^2} (c_- + c_+) \overline{\partial} j_z^a(w). \tag{2.27}$$

We can now show that the current j_z^a is a primary field of conformal weight one. We deduce from the previous computation the OPE between the stress-energy tensor and the current j_z^a, by expanding the operators on the right-hand side in the neigbourhood of the point z:

$$2c_1 T(w) j_z^a(z) = 2c_1 \frac{j_z^a(z)}{(w-z)^2} + 2c_1 \frac{\partial j_z^a(z)}{w-z} + 2(-c_1 + i(c_2 - g)(c_- + c_+)) \frac{\overline{z} - \overline{w}}{(z-w)^2} \overline{\partial} j_z^a(z) \tag{2.28}$$

Using the relation obtained in equation (2.19), we finally have

$$T(w) j_z^a(z) = \frac{j_z^a(z)}{(w-z)^2} + \frac{\partial j_z^a(z)}{w-z}, \tag{2.29}$$

which shows that the current j_z is a primary field of conformal dimension one. It can similarly be checked that $j_{\overline{z}}$ is a conformal primary of dimension zero.

The stress-energy tensor. We now want to compute the OPE between $T(z)$ and $T(w)$. This calculation relies on the preceeding calculation and on the double pole in the current-current operator product expansion. We get:

$$\begin{aligned} T(z)T(w) &= \frac{1}{2c_1} \lim_{:x \to w:} T(z) \left[j_{za}(x) j_z^a(w) \right] \\ &= \frac{1}{2c_1} \lim_{:x \to w:} \left[\left(\frac{j_{za}(x)}{(z-x)^2} + \frac{\partial j_{za}(x)}{z-x} \right) j_z^a(w) \right. \\ &\qquad \left. + j_{za}(x) \left(\frac{j_z^a(w)}{(z-w)^2} + \frac{\partial j_z^a(w)}{z-w} \right) \right] \end{aligned} \tag{2.30}$$

In the second line, only the regular terms in the remaining OPE's will contribute to the final result. In the first line, all the terms proportional to the structure constants disappear once again. We get:

$$\begin{aligned} T(z)T(w) = \frac{1}{2c_1} \lim_{:x \to w:} \Bigg[\Bigg(& \frac{c_1 \kappa_{ba} \kappa^{ba}}{(z-x)^2 (x-w)^2} + \frac{: j_{za} j_z^a : (w)}{(z-x)^2} - \frac{2c_1 \kappa_{ba} \kappa^{ba}}{(z-x)(x-w)^3} \\ & + \frac{: (\partial j_{za}) j_z^a : (w)}{z-x} \Bigg) + \frac{: j_{za} j_z^a : (w)}{(z-w)^2} + \frac{: j_{za} (\partial j_z^a) : (w)}{z-w} \Bigg] \end{aligned} \tag{2.31}$$

To take the limit, we expand all the functions of x in the neighbourhood of the point w and keep only the regular term:

$$T(z)T(w) = \frac{dim\ G}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}, \tag{2.32}$$

which proves that we have indeed a conformal algebra of central charge $c = dim\ G$. For supergroups, the relevant dimension is the superdimension (which is the self-contraction of the invariant metric).

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Summary. We have shown that a fairly generic current algebra leads to a conformal theory when the Killing form is zero. The corresponding stress-energy tensor is given by the Sugawara construction. The holomorphic current component is a conformal primary with respect to the holomorphic Virasoro algebra. The Sugawara energy-momentum tensor has a central charge equal to the superdimension of the supergroup.

We note that a supergroup with zero Coxeter number shares some features with a free theory. The central charge takes its naive value. The composite part of the Maurer-Cartan equation does not need to be renormalized. We will see other simplifications for these models further on.

3 Current algebra from supergroup current correlators

We now switch gears and consider a concrete model in which the generic analysis of two-dimensional current algebras of section 2 can be applied. We consider a conformal super-group sigma-model from the list given in [7]. Though we believe our analysis applies to the whole list, some facts that we use below have been proven explicitly only for the $PSL(n|n)$ models. We will calculate two-, three- and four-point functions of currents. Later we will infer the operator algebra of the currents from those correlation functions.

3.1 The model

We consider a supergroup non-linear sigma-model with standard kinetic term based on a bi-invariant metric on the supergroup. It is the principal chiral model on the supergroup. In addition we allow for a Wess-Zumino term. Therefore, we have two coupling constants, namely the coefficient of the kinetic term $1/f^2$ and the coupling constant k preceding the Wess-Zumino term. The action is:[10]

$$\begin{aligned} S &= S_{\rm kin} + S_{WZ} \\ S_{\rm kin} &= \frac{1}{16\pi f^2}\int d^2x Tr'[-\partial^\mu g^{-1}\partial_\mu g] \\ S_{WZ} &= -\frac{ik}{24\pi}\int_B d^3y \epsilon^{\alpha\beta\gamma} Tr'(g^{-1}\partial_\alpha g g^{-1}\partial_\beta g g^{-1}\partial_\gamma g) \end{aligned} \tag{3.1}$$

Using complex coordinates, and after taking the trace, the kinetic term becomes:

$$S_{\rm kin} = -\frac{1}{4\pi f^2}\int d^2z (\partial g g^{-1})_c(\bar{\partial} g g^{-1})^c. \tag{3.2}$$

The field g takes values in a supergroup.

[10] Our normalizations and conventions are mostly as in [23]. In particular we define the primed trace as $Tr'(t_a t_b) = 2\kappa_{ab}$ where κ_{ab} is normalized to be the Kronecker delta-function for a compact subgroup. The action is written in terms of real euclidean coordinates. We soon switch to complex coordinates via $z = x^1 + ix^2$. See [23] for further details. Starting in this section, we will no longer be careful in keeping track of the signs due to the bosonic or fermionic nature of the super Lie algebra generators. They can consistently be restored.

From the action we can calculate the classical currents associated to the invariance of the theory under left multiplication of the field g by a group element in G_L and right multiplication by a group element in G_R. The classical equations of motion for the model read:

$$\left(kf^2+1\right)\overline{\partial}J^a+\left(kf^2-1\right)\partial\left(g\overline{J}g^{-1}\right)^a=0, \tag{3.3}$$

where we have used the standard expressions for the left- and right-current at the Wess-Zumino-Witten point:[11]

$$J(z,\overline{z})=-k\partial g g^{-1} \qquad \text{and} \qquad \overline{J}(z,\overline{z})=kg^{-1}\overline{\partial}g\,. \tag{3.4}$$

The classical G_L currents are given by:

$$\begin{aligned} j_z &= -\frac{1}{2}\left(\frac{1}{f^2}+k\right)\partial g g^{-1}=\frac{(1+kf^2)}{2kf^2}J \\ j_{\overline{z}} &= -\frac{1}{2}\left(\frac{1}{f^2}-k\right)\overline{\partial} g g^{-1}=-\frac{(1-kf^2)}{2kf^2}(g\overline{J}g^{-1}). \end{aligned} \tag{3.5}$$

At the Wess-Zumino-Witten point $f^2=1/k$ the $\overline{z}$-component of the left-moving current becomes zero. As a consequence, the z-component J becomes holomorphic. A similar phenomenon happens at the other Wess-Zumino-Witten point ($f^2=-1/k$) for the anti-holomorphic component $g\overline{J}g^{-1}$. From now on, we will concentrate on the current j associated to the left action of the group. For future reference we note that the coefficients that relate the left current components to the derivative of the group element are (see section 2):

$$c_+=-\frac{(1+kf^2)}{2f^2} \qquad \text{and} \qquad c_-=-\frac{(1-kf^2)}{2f^2}\,. \tag{3.6}$$

3.2 Exact perturbation theory

Elegant arguments were given [6] for the exactness of low-order perturbation theory for the calculation of various observables in the supergroup model on $PSL(n|n)$. In particular, we will use these arguments to compute the (left) current-current two-point function exactly to all orders in perturbation theory using the free theory. Similarly we also compute the current three-point functions to all orders using perturbation theory up to first order in the structure constants. Below, we summarize some important facts that lead to these results [6].

The argument is essentially based on the special feature of the Lagrangian that all interaction vertices are proportional to (powers of) the structure constants as well as certain properties of the Lie superalgebra, which we now list:

- If structure constants are doubly contracted, the result is proportional to the Killing form, which is zero for the supergroups under consideration.
- The only invariant three-tensor is proportional to the structure constant and the only invariant two-tensor is the invariant metric.[12]

[11] At the Wess-Zumino-Witten point, the parameters satisfy the equation: $1/f^2=k$.

[12] These and the following are statements taken from [6]. A detailed proof is lacking. The condition of the uniqueness of the three-tensor can be relaxed to the condition that any invariant three-tensor contracted with the structure constants vanishes, which is a statement that has been proven in detail in an appendix to [11] for the particular case of the $psl(2|2)$ Lie superalgebra.

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- Traceless invariant 4-tensors made of structure constants and the invariant metric give zero when contracted with the structure constants over two indices [6].

Using all these facts, and by using a pictorial representation of the correlation functions, one can show that vacuum diagrams with at least one interaction vertex all vanish and that group invariant correlation functions can be computed in the free theory. However, in order to compute the OPEs in the theory of interest, we have to calculate the 2 and 3-point functions of the right-invariant currents J^a and $(g\overline{J}g^{-1})^a$, which are not fully invariant under the group action.

In [6], it is shown that a correlation function that is invariant under only the right group action, and which is a two-tensor under the left group action (or vice versa), can be computed by setting all structure constants to zero. Similarly, a correlation function that is invariant under the right group action, and a three-tensor under the left group action can be computed by taking into account only contributions with at most one structure constant. We will present the argument for the simplicity of the 3-point function in the next section.

In the following we also find it instructive to compute a four-point function to second order in the structure constants. In order to perform these calculations it is useful to expand the various terms in the action as well as the currents to second order in the structure constants.

Ingredients of perturbation theory. We gather all our ingredients expanded to second order in the structure constants. We use the conventions $g = e^A$ and $A = iA_a t^a$. For the left current components we obtain:

$$\begin{aligned} J &= -k\partial g g^{-1} = -k\left(\partial A + \frac{1}{2}[A,\partial A] + \frac{1}{3!}[A,[A,\partial A]] + O(f^3)\right) \\ &= -ki\left(\partial A_a - \frac{f^{bc}{}_a}{2}A_b\partial A_c + \frac{1}{6}f^a{}_{bc}f^c{}_{de}A^bA^d\partial A^e + O(f^3)\right)t^a \\ g\overline{J}g^{-1} &= k\overline{\partial}g g^{-1} = k\left(\overline{\partial}A + \frac{1}{2}[A,\overline{\partial}A] + \frac{1}{3!}[A,[A,\overline{\partial}A]] + O(f^3)\right) \\ &= +ki\left(\overline{\partial}A_a - \frac{f^{bc}{}_a}{2}A_b\overline{\partial}A_c + \frac{1}{6}f^a{}_{bc}f^c{}_{de}A^bA^d\overline{\partial}A^e + O(f^3)\right)t^a, \end{aligned} \tag{3.7}$$

where $O(f^3)$ indicates terms of third order or higher in the structure constants. The kinetic term and the Wess-Zumino term become:

$$\begin{aligned} S_{\text{kin}} &= \frac{1}{4\pi f^2}\int d^2z\left(\partial A^a\overline{\partial}A_a - \frac{1}{12}f^a{}_{bc}f_{aij}A^b\partial A^c A^i\overline{\partial}A^j + O(f^4)\right) \\ S_{WZ} &= -\frac{k}{12\pi}\int_C d^2z f^{abc}A_a\partial A_b\overline{\partial}A_c + O(f^3). \end{aligned} \tag{3.8}$$

The quadratic terms in the action give rise to the free propagators:

$$A_a(z,\overline{z})A_b(w,\overline{w}) = -f^2\kappa_{ab}\log\mu^2|z-w|^2, \tag{3.9}$$

where μ is an infrared regulator.

Two-point functions. Consider the Feynman diagrams with two external lines and pull out a structure constant where the external line enters. The rest of the diagram is now a blob with three external lines, with two of them contracted with the structure constant (the interaction strength). Now, from a group theoretic perspective, the three-spoked blob is also an invariant 3-tensor, which, by the properties itemized at the beginning of the section, is proportional to the structure constant. As a result, the whole graph is proportional to the metric times the Killing form: $f_{abc}f^{bc}_{d} = 2\check{h}g_{ad}$, which vanishes for the supergroups under consideration.

The two-point functions are therefore perturbatively exact when computed by setting all structure constants to zero and follows directly from the free propagator (3.9):

$$\begin{aligned}
\langle J^a(z,\overline{z})J^b(w,\overline{w})\rangle &= \frac{f^2k^2\kappa^{ab}}{(z-w)^2} \\
\langle (g\overline{J}g^{-1})^a(z,\overline{z})\left(g\overline{J}g^{-1}\right)^b(w,\overline{w})\rangle &= \frac{f^2k^2\kappa^{ab}}{(\overline{z}-\overline{w})^2} \\
\langle J^a(z,\overline{z})\left(g\overline{J}g^{-1}\right)^b(w,\overline{w})\rangle &= 2\pi f^2k^2\kappa^{ab}\delta^{(2)}(z-w).
\end{aligned} \tag{3.10}$$

Three-point functions. Consider the Feynman diagrams that contribute to

$$\langle J^a(z,\overline{z})J^b(w,\overline{w})J^c(x,\overline{x})\rangle\,.$$

In their evaluation, there are strucutre constants coming both from the expansion of the currents in (3.7) and also from the interaction vertices. We would like to argue that only those diagrams which contain a single structure constant contribute. In order to show this, consider pulling out one structure constant out of the vertex where the external line enters. The rest of the diagram can be thought of as a blob with four external lines and which has the group structure of a rank 4 invariant tensor. Contracting two of its indices, the resulting graph contains a structure constant inside and vanishes, following the same argument that allows to compute the two-point functions by setting all structure constants to zero.

We have now shown that the group structure of the four-spoked blob is that of a traceless rank 4 tensor. The full Feynman graph is evaluated by contracting a structure constant with this traceless rank-4 tensor. Using the special properties of the Lie superalgebra of $PSL(n|n)$ we listed earlier in the section, it is clear that such a term evaluates to zero. Thus, the three-point functions are perturbatively exact at first order in the structure constants. There are two non-trivial contributions to this calculation. We have one contribution coming from the term proportional to the structure constants in the expansion (3.7) of the current components, and one from the first order Wess-Zumino interaction (3.8).

Let us compute the first contribution for the JJJ three-point function in some detail:

$$\begin{aligned}
&\langle J^a(z,\overline{z})J^b(w,\overline{w})J^c(x,\overline{x})\rangle_1 = \\
&\qquad + ik^3\left\langle\left(\partial A^a(z,\overline{z}) - \frac{1}{2}f^{dea}A_d\partial A_e\right)\right. \\
&\qquad \left.\left(\partial A^b - \frac{1}{2}f^{deb}A_d\partial A_e\right)\left(\partial A^c - \frac{1}{2}f^{dec}A_d\partial A_e\right)\right\rangle
\end{aligned}$$

$$
\begin{aligned}
&= -ik^3 f^4 \frac{1}{2}\left((+)\frac{f^{abc}}{(z-x)(w-x)^2} + (+)\frac{f^{bac}}{(w-x)(z-x)^2} + cyclic \right) \\
&= -i\frac{1}{2}k^3 f^4 f^{abc}\frac{z-w}{(z-x)^2(w-x)^2} + cyclic \\
&= -i\frac{3}{2}k^3 f^4 f^{abc}\frac{1}{(z-w)(w-x)(x-z)}\,. \qquad (3.11)
\end{aligned}
$$

The Wess-Zumino contribution is:[13]

$$
\begin{aligned}
\left\langle J^a(z,\overline{z})J^b(w,\overline{w})J^c(x,\overline{x})\right\rangle_2 &= +\,ik^4\left\langle \partial A^a(z,\overline{z})\partial A^b(w,\overline{w})\partial A^c(x,\overline{x})\times \right. \\
&\qquad \left. \frac{1}{12\pi}\int_C d^2y f^{deg} A_g \partial A_d \overline{\partial} A_e(y,\overline{y})\right\rangle \\
&= +ik^4 f^6 \frac{1}{2} f^{abc}\frac{1}{(z-w)(w-x)(x-z)}. \qquad (3.12)
\end{aligned}
$$

Adding the two contributions we get the three-point function:

$$
\langle J^a(z,\overline{z})J^b(w,\overline{w})J^c(x,\overline{x})\rangle \;=\; -i\frac{1}{2}k^3 f^4(3-kf^2)f^{abc}\frac{1}{(z-w)(w-x)(x-z)}. \qquad (3.13)
$$

A quick check on the calculation is that it matches the known three-point function at the Wess-Zumino-Witten point, where it can be evaluated using the holomorphy of the currents. All other left current three-point functions can be computed analogously. They are (up to contact terms):

$$
\begin{aligned}
\langle J^a(z,\overline{z})J^b(w,\overline{w})J^c(x,\overline{x})\rangle &= \left(\frac{3}{2}-\frac{kf^2}{2}\right)\frac{-ik^3f^4f^{abc}}{(z-w)(w-x)(x-z)} \\
\langle J^a(z,\overline{z})J^b(w,\overline{w})(g\overline{J}g^{-1})^c(x,\overline{x})\rangle &= \left(\frac{1}{2}-\frac{kf^2}{2}\right)\frac{-ik^3f^4f^{abc}(\overline{z}-\overline{w})}{(z-w)^2(\overline{x}-\overline{w})(\overline{x}-\overline{z})} \\
\langle (g\overline{J}g^{-1})^a(z,\overline{z})(g\overline{J}g^{-1})^b(w,\overline{w})J^c(x,\overline{x})\rangle &= \left(\frac{1}{2}+\frac{kf^2}{2}\right)\frac{+ik^3f^4f^{abc}(z-w)}{(\overline{z}-\overline{w})^2(x-w)(x-z)} \\
\langle (g\overline{J}g^{-1})^a(z,\overline{z})(g\overline{J}g^{-1})^b(w,\overline{w})(g\overline{J}g^{-1})^c(x,\overline{x})\rangle &= \left(\frac{3}{2}+\frac{kf^2}{2}\right)\frac{+ik^3f^4f^{abc}}{(\overline{z}-\overline{w})(\overline{w}-\overline{x})(\overline{x}-\overline{z})}\,. \qquad (3.14)
\end{aligned}
$$

Coincidence limit and operator product expansions. When we take coincidence limits of the three-point functions, we expect to be able to replace the product of two operators by their operator product expansion. Using the general form of the current-current operator product expansions, and the exact two-point functions, we can infer from the above three-point functions a proposal for the current-current operator product expansions. Up to contact terms the two- and three-point functions can be reproduced in their

[13] A minus sign arises from expanding e^{-S} to first order.

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coincidence limits by the OPEs:[14]

$$
\begin{aligned}
J^a(z,\bar z)J^b(w,\bar w) \sim\ & \frac{k^2f^2\kappa^{ab}}{(z-w)^2}+kf^2\left(\frac{3}{2}-\frac{kf^2}{2}\right)if^{ab}{}_c\frac{J^c(w)}{z-w}\\
&+(-kf^2)\left(\frac{1}{2}-\frac{kf^2}{2}\right)if^{ab}{}_c\frac{\bar z-\bar w}{(z-w)^2}\left(g\bar J g^{-1}\right)^c(w)+\cdots\\
J^a(z,\bar z)(g\bar Jg^{-1})^b(w,\bar w) \sim\ & 2\pi k^2f^2\kappa^{ab}\delta^{(2)}(z-w)+kf^2\Big(\frac{1}{2}+\frac{kf^2}{2}\Big)if^{ab}{}_c\frac{(g\bar Jg^{-1})^c(w)}{z-w}\\
&-kf^2\left(\frac{1}{2}-\frac{kf^2}{2}\right)if^{ab}{}_c\frac{1}{\bar z-\bar w}J^c(w)+\cdots\\
(g\bar Jg^{-1})^a(z,\bar z)(g\bar Jg^{-1})^b(w,\bar w) \sim\ & \frac{k^2f^2\kappa^{ab}}{(\bar z-\bar w)^2}-kf^2\left(\frac{3}{2}+\frac{kf^2}{2}\right)if^{ab}{}_c\frac{(g\bar Jg^{-1})^c(w)}{\bar z-\bar w}\\
&+kf^2\left(\frac{1}{2}+\frac{kf^2}{2}\right)if^{ab}{}_c\frac{z-w}{(\bar z-\bar w)^2}J^c(w)+\cdots
\end{aligned}
\tag{3.15}
$$

When we normalize the currents as in (3.5) to agree with section 2, we find the following OPEs:

$$
\begin{aligned}
j^a_z(z)j^b_w(w) \sim\ & \frac{(1+kf^2)^2\kappa^{ab}}{4f^2(z-w)^2}+\frac{i}{4}(1+kf^2)(3-kf^2)f^{ab}{}_c\frac{j^c_w(w)}{z-w}\\
&+\frac{i}{4}(1+kf^2)^2f^{ab}{}_c\frac{\bar z-\bar w}{(z-w)^2}j^c_{\bar w}(w)+\cdots\\
j^a_{\bar z}(z)j^b_{\bar w}(w) \sim\ & \frac{(1-kf^2)^2\kappa^{ab}}{4f^2\,(\bar z-\bar w)^2}+\frac{i}{4}(1-kf^2)(3+kf^2)\frac{f^{ab}{}_c j^c_{\bar w}(w)}{\bar z-\bar w}\\
&+\frac{i}{4}(1-kf^2)^2\frac{(z-w)\,f^{ab}{}_c j^c_w(w)}{(\bar z-\bar w)^2}+\cdots\\
j^a_z(z)j^b_{\bar w}(w) \sim\ & -\frac{2\pi}{4f^2}(1+kf^2)(1-kf^2)\kappa^{ab}\delta^{(2)}(z-w)+\frac{i(1+kf^2)^2}{4}\frac{f^{ab}{}_c j^c_{\bar w}(w)}{z-w}\\
&+\frac{i(1-kf^2)^2}{4}\frac{f^{ab}{}_c j^c_w(w)}{\bar z-\bar w}+\cdots
\end{aligned}
\tag{3.16}
$$

We can read from these formulas the coefficients of the generic current algebra (2.21):

$$
\begin{aligned}
c_1 &= \frac{(1+kf^2)^2}{4f^2} \qquad c_2=\frac{i}{4}(1+kf^2)(3-kf^2)\\
c_3 &= \frac{(1-kf^2)^2}{4f^2} \qquad c_4=\frac{i}{4}(1-kf^2)(3+kf^2)\\
\text{and}\quad g &= \frac{i}{2}(1+kf^2)(1-kf^2)\,.
\end{aligned}
\tag{3.17}
$$

We note that the coefficients automatically satisfy the extra constraints (2.19) and (2.20) one gets by requiring consistency of the current algebra with the Maurer-Cartan equation.

[14] From now on we will no longer always make explicit the fact that all the operators depend both on the holomorphic as well as the anti-holomorphic coordinate at a generic point in the moduli space.

A four-point function. From the three-point functions, we conclude that the coefficient g is non-zero. We have argued in section 2 that that is associated to the appearance of logarithms in the regular term of the $j_z j_{\overline{z}}$ operator product expansion. We would like to check the coefficient of the logarithm more directly in a perturbative calculation. For that purpose it is sufficient to study a four-point function at second order in the structure constants. The computation will be exact at that order. In particular we want to compute the four-point function:

$$\langle J^{[a}(z)(g\overline{J}g^{-1})^{b]}(w)J^c(x)(g\overline{J}g^{-1})^d(y)\rangle_{O(f^2)}, \tag{3.18}$$

at second order in the structure constants, and in the $z \to w$ limit. We anti-symmetrized in the a and b indices (and weighted each term with a factor $1/2$). In the coincidence limit, logarithms appear in the regular terms in the OPE between J and $(g\overline{J}g^{-1})$ and they will give non-zero contribution to the four-point functions. In our calculation we focus on the terms proportional to $\log|z-w|^2$ (and which are not contact terms).

We distinghuish the following contributions at this order. We can expand a single current to second order, and compute in the free theory. We can expand two currents to first order and compute in the free theory. We can add one linear Wess-Zumino interaction term and expand one current to first order. Or we can add two linear Wess-Zumino interaction terms and take only the leading terms in the currents. Finally, we can add one quadratic principal chiral model interaction term, and treat the currents at zeroth order.

We found the following results. The second order term in a current cannot give rise to logarithmic contributions. Two currents at first order can be contracted to give a logarithm. It is easy to see that there are few contributions to the terms of interest, and they give:

$$-\frac{k^4 f^6}{8} f^{abe} f^{cd}{}_e \log \mu^2|z-w|^2 \frac{1}{(z-x)^2(\overline{w}-\overline{y})^2} \tag{3.19}$$

The term linear in the Wess-Zumino interaction term gives no contribution. The term arising from the quartic interaction term in the principal chiral model doubles the previous non-zero term. The Wess-Zumino term squared gives a contribution of a different type equal to:

$$+\frac{k^6 f^{10}}{4} \log \mu^2|z-w|^2 f^{abe} f^{cd}{}_e \frac{1}{(z-x)^2(\overline{w}-\overline{y})^2}. \tag{3.20}$$

The latter contribution is the hardest to calculate. It consists of the order of 216 free field contractions, which exhibit a lot of symmetries. Some logarithmic terms in the double integrals over the points of interaction need to be evaluated, but all of these integrals are straightforwardly performed using partial integrations and other elementary techniques. The tedious but elementary calculation leads to the above result. In total we get, at second order in the structure constants, and only regarding the logarithmic contribution in z, w:

$$\langle J^{[a}(z)(g\overline{J}g^{-1})^{b]}(w)J^c(x)(g\overline{J}g^{-1})^d(y)\rangle_{O(f^2),log} = \\ -\frac{1}{4}k^4 f^6(1-k^2f^4) f^{abe} f^{cd}{}_e \log \mu^2|z-w|^2 \frac{1}{(z-x)^2(\overline{w}-\overline{y})^2}. \tag{3.21}$$

Using the relative normalization between the J's and the j's, we find that

$$\langle j_z^{[a}(z) j_{\overline{w}}^{b]}(w) j_x^c(x) j_{\overline{y}}^d(y)\rangle_{O(f^2),log} =$$
$$-\frac{1}{64f^2}(1+kf^2)^2(1-kf^2)(1-k^2f^4)f^{abe}f^{cd}{}_e \log \mu^2|z-w|^2 \frac{1}{(z-x)^2(\overline{w}-\overline{y})^2}\,. \quad (3.22)$$

Let us see how to use this result to check the coefficient g in the operator product expansion. From the expressions for the operator product algebra (2.21) and from the exact three-point functions (3.14), we find that the logarithmic term in the normalized four-point function in the coincidence limit $z \to w$ is:

$$\begin{aligned}
\langle j_z^a j_{\overline{w}}^b j_x^c j_{\overline{y}}^d\rangle &\approx -\frac{g}{4} f^{ab}{}_e \log \mu^2|z-w|^2 \langle (\partial j_{\overline{w}}^e - \overline{\partial} j_w^e) j_x^c j_{\overline{y}}^d\rangle \\
&\approx +\frac{g}{4}\frac{1}{8k^3f^6}(1+kf^2)(1-kf^2)f^{ab}{}_e \log \mu^2|z-w|^2 \\
&\quad \Bigg(-\partial_w(1-kf^2)ik^3f^4f^{edc}\frac{1}{2}(1+kf^2)\frac{w-y}{(\overline{w}-\overline{y})^2(x-w)(x-y)} \\
&\quad -\overline{\partial}_w(1+kf^2)f^{ecd}(-)ik^3f^4\frac{1}{2}(1-kf^2)\frac{\overline{w}-\overline{x}}{(w-x)^2(\overline{y}-\overline{w})(\overline{y}-\overline{x})}\Bigg) \\
&\approx +i\frac{g}{32f^2}(1+kf^2)^2(1-kf^2)^2f^{ab}{}_e f^{ecd} \\
&\quad \log \mu^2|z-w|^2 \frac{1}{(w-x)^2(\overline{w}-\overline{y})^2}.
\end{aligned} \quad (3.23)$$

We recall the value for g:

$$g = \frac{i}{2}(1+kf^2)(1-kf^2), \quad (3.24)$$

so the operator algebra and the three-point functions predict:

$$\begin{aligned}
\langle j_z^a j_{\overline{w}}^b j_x^c j_{\overline{y}}^d\rangle \approx -\frac{1}{64f^2}&(1+kf^2)^2(1-kf^2)^2(1-k^2f^4)f^{ab}{}_e f^{ecd} \\
&\log \mu^2|z-w|^2 \frac{1}{(w-x)^2(\overline{w}-\overline{y})^2}.
\end{aligned} \quad (3.25)$$

The prediction is matched by our perturbative calculation of the four-point function. Moreover, since the coefficient g is fixed to all orders by the calculation of the three-point function, our result at second order in the structure constants is exact. The calculation is a good consistency check on the correlators and operator product expansions. The full four-point function is a function of the cross ratio of the four insertion points, in which the regulator μ drops out. We note also that the appearance of logarithms in four-point functions of operators that differ by an integer in their conformal dimension is generic. In our case, the scale μ must appear in the operator product expansion because we lifted space-time fermionic zeromodes [6]. These in turn are linked to the non-diagonalizable nature of the scaling operator in sigma-models on supergroups [24].

3.3 Summary of the current algebra

We summarize the current algebra for the left group action:

$$
\begin{aligned}
j_z^a(z)j_z^b(w) &= \frac{(1+kf^2)^2\kappa^{ab}}{4f^2(z-w)^2} + \frac{i}{4}(1+kf^2)(3-kf^2)\frac{f^{ab}{}_c\, j_z^c(w)}{z-w} \\
&\quad + \frac{i}{4}(1+kf^2)^2\frac{\overline{z}-\overline{w}}{(z-w)^2}f^{ab}{}_c\, j_{\overline{z}}^c(w) + :j_z^a j_z^b:(w) \\
j_z^a(z)j_{\overline{z}}^b(w) &= -\frac{\pi}{4f^2}(1+kf^2)(1-kf^2)\kappa^{ab}\delta^{(2)}(z-w) + \frac{i(1+kf^2)^2}{4}\frac{f^{ab}{}_c\, j_{\overline{z}}^c(w)}{z-w} \\
&\quad + \frac{i(1-kf^2)^2}{4}\frac{f^{ab}{}_c\, j_z^c(w)}{\overline{z}-\overline{w}} - \frac{i}{8}(1-k^2f^4)f^{ab}{}_c \log|z-w|^2\left(\partial j_{\overline{z}}^c(w) - \overline{\partial} j_z^c(w)\right) \\
&\quad + :j_z^a j_{\overline{z}}^b:(w) \\
j_{\overline{z}}^a(z)j_{\overline{z}}^b(w) &= \frac{(1-kf^2)^2\kappa^{ab}}{4f^2(\overline{z}-\overline{w})^2} + \frac{i}{4}(1-kf^2)(3+kf^2)\frac{f^{ab}{}_c\, j_{\overline{w}}^c(w)}{\overline{z}-\overline{w}} \\
&\quad + \frac{i(1-kf^2)^2}{4}\frac{z-w}{(\overline{z}-\overline{w})^2}f^{ab}{}_c\, j^c(w) + :j_{\overline{z}}^a j_{\overline{z}}^b:(w)\,.
\end{aligned} \tag{3.26}
$$

The current algebra for the right group action can be obtained through the combined operation $g \to g^{-1}$ and worldsheet parity P, which is a symmetry of the model

4 Conformal perturbation theory

In this section we study Wess-Zumino-Witten models with a perturbed kinetic term, both for its intrinsic interest and as a tractable example of conformal perturbation theory. For a general group manifold, the deformed model becomes non-conformal. For supergroup manifolds with vanishing Killing form, the models remain conformal. In the earlier sections, we have computed, using the exact two- and three-point functions, the current algebra of the deformed theory. In this section, we re-derive these results using conventional conformal perturbation theory. This will be a consistency check of the deformed conformal current algebra we have obtained in equation (3.26).

4.1 The current algebra in the Wess-Zumino-Witten model

We first review the chiral current algebra of the Wess-Zumino-Witten model. We recall the action

$$S_{WZW} = \frac{k}{16\pi}\int d^2x Tr'[-\partial^\mu g^{-1}\partial_\mu g] + k\Gamma \tag{4.1}$$

where Γ is the Wess-Zumino term, and the field $g(z,\overline{z})$ takes values in a (super)group G. The model has a global $G_L \times G_R$ invariance by left and right multiplication of the group element. The currents associated to these symmetries are (in complex coordinates):

$$J(z) = -k\partial g g^{-1} \qquad \text{and} \qquad \overline{J}(\overline{z}) = kg^{-1}\overline{\partial} g\,. \tag{4.2}$$

The right-invariant current $J(z)$ is holomorphic and generates the left-action of the group G_L. The left-invariant anti-holomorphic current $\overline{J}$ generates the right translation by a group element. The components of the current J satisfy the OPE:

$$J^a(z)J^b(w) \sim \frac{k\kappa^{ab}}{(z-w)^2} + if^{ab}{}_c \frac{J^c(w)}{z-w}, \tag{4.3}$$

and the components of the current $\overline{J}(\overline{z})$ satisfy the same OPE, with anti-holomorphic coordinates instead of holomorphic ones. In particular, in our conventions the pole term keeps the same sign. These currents generate a large chiral affine current algebra whose existence is useful in solving the model via the Knizhnik-Zamolodchikov equations.

4.2 Perturbation of the kinetic term: classical analysis

We are interested in the following marginal deformation of the Wess-Zumino-Witten model:

$$S = S_{WZW} + \frac{\lambda}{4\pi k}\int d^2z\, \Phi(z,\overline{z})\,. \tag{4.4}$$

where

$$\Phi = \frac{1}{2}(: J^c(g\overline{J}g^{-1})_c : + : (g\overline{J}g^{-1})_c J^c :)\,. \tag{4.5}$$

In other words, we perturb the kinetic term by multiplying it with a factor $1+\lambda$. Comparing the action with the action in the earlier section, we find that λ is related to the kinetic coefficient f defined in the previous section by the relation

$$\frac{1}{f^2} = k(1+\lambda)\,. \tag{4.6}$$

We note that, analogous to the composite operator that appeared in the Maurer-Cartan equation, we have chosen a symmetric combination of the product of J and $g\overline{J}g^{-1}$ operators to represent the marginal operator in the quantum theory.

4.3 The current-current operator product expansions

In this subsection we compute the correction to the holomorphic current-current operator product expansion induced by the perturbation of the kinetic term of the Wess-Zumino-Witten model for a simple (super) Lie algebra. In order to perform the calculation we require the OPEs between the currents J and $g\overline{J}g^{-1}$ at the WZW point. These are obtained by requiring that the Maurer-Cartan equation holds in the quantum WZW model, as shown in appendix B: we compute the OPE of the current J with the Maurer-Cartan equation for a generic value of the dual Coxeter number and demand that it vanish. This constraint leads to the operator product expansion

$$J^a(z)(g\overline{J}g^{-1})^b(w,\overline{w}) = 2\pi k\kappa^{ab}\delta^{(2)}(z-w) + if^{ab}{}_c\frac{(g\overline{J}g^{-1})^c(w,\overline{w})}{z-w} + : J^a(g\overline{J}g^{-1})^b : (w,\overline{w})\,. \tag{4.7}$$

A similar demand on contact terms and the most singular terms in the OPE of $(g\overline{J}g^{-1})$ with the Maurer-Cartan equation leads to the OPE

$$\begin{aligned}(g\overline{J}g^{-1})^a(z,\overline{z})(g\overline{J}g^{-1})^b(w,\overline{w}) = {} & \frac{k\kappa^{ab}}{(\overline{z}-\overline{w})^2} + if^{ab}{}_c\frac{J^c(w)(z-w)}{(\overline{z}-\overline{w})^2} - 2if^{ab}{}_c\frac{(g\overline{J}g^{-1})^c(w,\overline{w})}{\overline{z}-\overline{w}} \\ & + : (g\overline{J}g^{-1})^a(g\overline{J}g^{-1})^b : (w,\overline{w})\,. \end{aligned} \tag{4.8}$$

A general discussion of higher order corrections to operator product expansions is given in appendix A. Here we focus on applying the discussion to the case of a supergroup with vanishing Killing form. We compute the corrections induced by the exactly marginal perturbation to the $J^a J^b$ OPE. The nth order correction is denoted by

$$(JJ)^{ab}_n(z-w,w) = \left[J^a(z,\overline{z}) \frac{(-\lambda)^n}{(4\pi k)^n n!} \prod_{i=1}^{n} \int d^2x_i \Phi(x_i,\overline{x}_i) \right] J^b(w,\overline{w}) \tag{4.9}$$

where the square bracket means that we have to contract $J^a(z,\overline{z})$ with all the integrated operators before we contract it with $J^b(w,\overline{w})$. We define $H^a_n(z,\overline{z})$ to be this complete contraction:

$$H^a_n(z,\overline{z}) = \left[J^a(z,\overline{z}) \frac{1}{(4\pi k)^n n!} \prod_{i=1}^{n} \int d^2x_i \Phi(x_i,\overline{x}_i) \right] . \tag{4.10}$$

One can similarly define another contraction, with J replaced by the current $(g\overline{J}g^{-1})^a$:

$$\overline{H}^a_n(z,\overline{z}) = \left[(g\overline{J}g^{-1})^a(z,\overline{z}) \frac{1}{(4\pi k)^n n!} \prod_{i=1}^{n} \int d^2x_i \Phi(x_i,\overline{x}_i) \right] . \tag{4.11}$$

The basic building blocks we need to carry out this computation are the OPE of the currents J^a and $(g\overline{J}g^{-1})^a$ with the marginal operator Φ. As we will see, once these OPEs are obtained, the nth order correction can be obtained by a process of iteration. Here, we list the two OPEs of interest and refer the reader to appendix B for details.

$$\begin{aligned} J^a(w) \int d^2x \Phi(x,\overline{x}) &\sim k \int d^2x \frac{(g\overline{J}g^{-1})^a(w,\overline{w})}{(w-x)^2} + 2\pi k J^a(w) \\ (g\overline{J}g^{-1})^a(w,\overline{w}) \int d^2x \Phi(x,\overline{x}) &\sim 6\pi k (g\overline{J}g^{-1})^a(w,\overline{w}) - k \int d^2x \frac{J^a(x,\overline{x})}{(\overline{w}-\overline{x})^2} . \end{aligned} \tag{4.12}$$

With these basic OPEs, let us contract the current with one of the integrated marginal operators:

$$\begin{aligned} H^a_n(z,\overline{z}) =& \frac{n}{(4\pi k)^n n!} \int d^2x \left(\frac{k(g\overline{J}g^{-1})^a(x,\overline{x})}{(z-x)^2} + 2\pi k \delta^{(2)}(z-x) J^a(x,\overline{x}) \right) \\ & \prod_{i=1}^{n-1} \int d^2x_i \Phi(x_i,\overline{x}^i) \\ =& \frac{1}{4\pi} \int d^2x \left(\frac{\overline{H}^a_{n-1}(x,\overline{x})}{(z-x)^2} \right) + \frac{1}{2} H^a_{n-1}(z,\overline{z}) \end{aligned} \tag{4.13}$$

One can do a similar operation on $\overline{H}^a_n$ and we get

$$\begin{aligned} \overline{H}^a_n(z,\overline{z}) &= \frac{n}{(4\pi k)^n n!} \left[6\pi k (g\overline{J}g^{-1})^a(z,\overline{z}) - \int d^2x \frac{kJ^a(x,\overline{x})}{(\overline{z}-\overline{x})^2} \right] \prod_{i=1}^{n-1} \int d^2x_i \Phi(x_i,\overline{x}^i) \\ &= \frac{3}{2} \overline{H}^a_{n-1}(z,\overline{z}) - \frac{1}{4\pi} \int d^2x \left(\frac{H^a_{n-1}(x,\overline{x})}{(\overline{z}-\overline{x})^2} \right) . \end{aligned} \tag{4.14}$$

These are coupled recursion relations for H_n^a and $\overline{H}_n^a$ subject to the initial conditions:

$$\begin{aligned} H_0^a(z,\overline{z}) &= J^a(z,\overline{z}) \\ \overline{H}_0^a(z,\overline{z}) &= (g\overline{J}g^{-1})^a(z,\overline{z}) \end{aligned} \tag{4.15}$$

These recursion relations have the following solutions:

$$\begin{aligned} H_n^a(z,\overline{z}) &= \left(1-\frac{n}{2}\right) J^a(z,\overline{z}) + \frac{n}{2}\frac{1}{2\pi}\int d^2x \frac{(g\overline{J}g^{-1})^a(x,\overline{x})}{(z-x)^2} \\ \overline{H}_n^a(z,\overline{z}) &= \left(1+\frac{n}{2}\right)(g\overline{J}g^{-1})^a(z,\overline{z}) - \frac{n}{2}\frac{1}{2\pi}\int d^2x \frac{J^a(x,\overline{x})}{(\overline{z}-\overline{x})^2}\,. \end{aligned} \tag{4.16}$$

In particular we deduce that

$$\begin{aligned} (JJ)_n^{ab}(z-w,w) =&(-\lambda)^n J^b(w,\overline{w}) \left[\left(1-\frac{n}{2}\right) J^a(z,\overline{z}) + \frac{n}{2}\frac{1}{2\pi}\int d^2x \frac{(g\overline{J}g^{-1})^a(x,\overline{x})}{(z-x)^2}\right] \\ =&(-\lambda)^n \left[\left(1-\frac{n}{2}\right)\left(\frac{k\kappa^{ab}}{(z-w)^2} + if^{abc}\frac{J^c(w)}{z-w}\right)\right. \\ &\left. + \frac{n}{2}\frac{1}{2\pi}\int d^2x \frac{1}{(z-x)^2}\left(2\pi k\kappa^{ab}\delta^{(2)}(z-x) + if^{abc}\frac{(g\overline{J}g^{-1})^c(w,\overline{w})}{x-w}\right)\right] \\ =&(-\lambda)^n \left[\frac{k\kappa^{ab}}{(z-w)^2} + if^{abc}\left(1-\frac{n}{2}\right)\frac{J^c(w)}{z-w}\right. \\ &\left. +i\frac{n}{2}f^{abc}(g\overline{J}g^{-1})^c(w,\overline{w})\frac{\overline{z}-\overline{w}}{(z-w)^2} + \cdots\right]. \end{aligned} \tag{4.17}$$

We can now sum the perturbative series in λ. We get the OPE in the perturbed theory:

$$\begin{aligned} J^a(z,\overline{z})J^b(w,\overline{w}) =& \sum_{n=0}^{\infty}(-\lambda)^n (JJ)_n^{ab}(z-w,w) \\ =&\frac{1}{1+\lambda}\frac{k\kappa^{ab}}{(z-w)^2} + \frac{2+3\lambda}{2(1+\lambda)^2}if^{abc}\frac{J^c(w)}{z-w} \\ &- \frac{\lambda}{2(1+\lambda)^2}if^{abc}(g\overline{J}g^{-1})^c(w,\overline{w})\frac{\overline{z}-\overline{w}}{(z-w)^2} \end{aligned}$$

Using the map $(kf^2)^{-1} = 1+\lambda$, one can check that this coincides with the OPE in equation (3.15). With the same techniques we can also compute the corrections to the $J^a(g\overline{J}g^{-1})^b$ and $(g\overline{J}g^{-1})^a(g\overline{J}g^{-1})^b$ OPEs. We get the results

$$\begin{aligned} J^a(z,\overline{z})(g\overline{J}g^{-1})^b(w,\overline{w}) =& \frac{1}{1+\lambda}2\pi k\kappa^{ab}\delta^{(2)}(z-w) + if^{abc}\frac{2+\lambda}{2(1+\lambda)^2}\frac{(g\overline{J}g^{-1})^c(w,\overline{w})}{z-w} \\ &-if^{abc}\frac{\lambda}{2(1+\lambda)^2}\frac{J^c(w)}{\overline{z}-\overline{w}} + \cdots \end{aligned} \tag{4.18}$$

$$\begin{aligned} (g\overline{J}g^{-1})^a(z,\overline{z})(g\overline{J}g^{-1})^b(w,\overline{w}) =& \frac{1}{1+\lambda}\frac{k\kappa^{ab}}{(\overline{z}-\overline{w})^2} - \frac{4+3\lambda}{2(1+\lambda)^2}\frac{if^{abc}(g\overline{J}g^{-1})^c(w,\overline{w}}{\overline{z}-\overline{w}} \\ &+\frac{2+\lambda}{2(1+\lambda)^2}\frac{(z-w)if^{abc}J^c(w)}{(\overline{z}-\overline{w})^2} + \cdots \end{aligned} \tag{4.19}$$

Again this matches with the OPE obtained in equation (3.15).

Summary. In this section, we have shown by resumming conformal perturbation theory that the deformed current algebra we obtain this way matches the algebra obtained in section 3 through the calculation of 2- and 3-point functions to all orders in perturbation theory.

5 The current algebra on the cylinder

In this section we consider the sigma-model on a cylinder and Fourier decompose the current algebra. The representation in terms of Fourier modes is often more conventient. To put the algebra on a cylinder, we first compute the operator algebra on the plane, and then compactify the plane. We consider the complex plane $z = \sigma - i\tau$ and consider τ as time and σ as the spatial coordinate. Denoting the currents as $j_\mu(\sigma, \tau)$, the commutator of equal-time operators is defined to be the limit of the difference of time-ordered operators (evaluated at $\tau = 0$)

$$[j^a_\mu(\sigma, 0), j^b_\nu(0, 0)] = \lim_{\epsilon\to 0}\left(j^a_\mu(\sigma, i\epsilon)j^b_\nu(0, 0) - j^b_\nu(0, i\epsilon)j^a_\mu(\sigma, 0)\right) . \tag{5.1}$$

Using this definition, let us compute the commutators for the holomorphic component of the current (we suppress the $\tau = 0$ argument within the currents in what follows):

$$\begin{aligned}[j^a_z(\sigma), j^b_z(0)] &= \lim_{\epsilon\to 0}\Bigg\{\frac{c_1\kappa^{ab}}{(\sigma - i\epsilon)^2} + f^{ab}{}_c\left(\frac{c_2}{\sigma - i\epsilon}j^c_z(0) + (c_2 - g)\frac{\sigma + i\epsilon}{(\sigma - i\epsilon)^2}j^c_{\bar z}(0)\right) \\ &\qquad -\frac{c_1\kappa^{ab}}{(\sigma + i\epsilon)^2} - f^{ab}{}_c\left(\frac{c_2}{-\sigma - i\epsilon}j^c_z(\sigma) + (c_2 - g)\frac{-\sigma + i\epsilon}{(\sigma + i\epsilon)^2}j^c_{\bar z}(\sigma)\right)\Bigg\} + \ldots \\ &= -2\pi i c_1\delta'(\sigma)\kappa^{ab} + 2\pi i c_2\delta(\sigma)f^{ab}{}_c j^c_z(0) + 2\pi i(c_2 - g)\delta(\sigma)f^{ab}{}_c j^c_{\bar z}(0) + \ldots\end{aligned} \tag{5.2}$$

where we used

$$\begin{aligned}\lim_{\epsilon\to 0}\frac{1}{\sigma - i\epsilon} - \frac{1}{\sigma + i\epsilon} &= 2\pi i\delta(\sigma) \\ \lim_{\epsilon\to 0}\frac{1}{(\sigma - i\epsilon)^2} - \frac{1}{(\sigma + i\epsilon)^2} &= -2\pi i\delta'(\sigma) \\ \lim_{\epsilon\to 0}\frac{\sigma + i\epsilon}{(\sigma - i\epsilon)^2} - \frac{\sigma - i\epsilon}{(\sigma + i\epsilon)^2} &= 2\pi i\delta(\sigma).\end{aligned} \tag{5.3}$$

For other components we find:

$$\begin{aligned}[j^a_{\bar z}(\sigma), j^b_{\bar z}(0)] &= +2\pi i c_3\delta'(\sigma)\kappa^{ab} - 2\pi i c_4\delta(\sigma)f^{ab}{}_c j^c_{\bar z}(0) - 2\pi i(c_4 - g)\delta(\sigma)f^{ab}{}_c j^c_z(0) \\ [j^a_z(\sigma), j^b_{\bar z}(0)] &= -2\pi i(c_4 - g)\delta(\sigma)f^{ab}{}_c j^c_z(0) + 2\pi i(c_2 - g)\delta(\sigma)f^{ab}{}_c j^c_{\bar z}(0)\,.\end{aligned} \tag{5.4}$$

It is now straightforward to compactify $\sigma \equiv \sigma + 2\pi$ and Fourier decompose the operator algebra on the cylinder using:

$$\begin{aligned}j_z &= +i\sum_{n\in Z} e^{-in\sigma} j_{z,n} \\ j_{\bar z} &= -i\sum_{n\in Z} e^{-in\sigma} j_{\bar z,n} \\ \delta(\sigma) &= \frac{1}{2\pi}\sum_{n\in Z} e^{in\sigma}.\end{aligned} \tag{5.5}$$

We find:

$$\begin{aligned}
[j^a_{z,n}, j^b_{z,m}] &= c_1\kappa^{ab} n\delta_{n+m,0} + c_2 f^{ab}{}_c j^c_{z,n+m} - (c_2 - g) f^{ab}{}_c j^c_{\bar{z},n+m} \\
[j^a_{\bar{z},n}, j^b_{\bar{z},m}] &= -c_3\kappa^{ab} n\delta_{n+m,0} + c_4 f^{ab}{}_c j^c_{\bar{z},n+m} - (c_4 - g) f^{ab}{}_c j^c_{z,n+m} \\
[j^a_{z,n}, j^b_{\bar{z},m}] &= (c_4 - g) f^{ab}{}_c j^c_{z,n+m} + (c_2 - g) f^{ab}{}_c j^c_{\bar{z},n+m}.
\end{aligned} \tag{5.6}$$

We can check the validity of the Jacobi identity, which follows from the validity of the Maurer-Cartan equation along with the Jacobi identity for the Lie algebra of G.

Conserved charges

We note that the current one-form satisfies the conservation equation $d * j = 0$, and that therefore the integral of the time-component of the current over the spatial circle is conserved in time. The corresponding charges are easily determined to be the sum of the zero-modes of the current algebra. They generate the Lie algebra of G. We recall that the group action generated by these charges corresponds to the left group action G_L, and that there is an analogous right group action G_R.

Kac-Moody subalgebra. Let us consider the combination of the currents $j^a_z - j^a_{\bar{z}}$ and compute the commutation relations of its modes with themselves. Using the above basic commutation relations, we find

$$[(j^a_{z,n} + j^a_{\bar{z},n}), (j^b_{z,m} + j^b_{\bar{z},m})] = (c_1 - c_3)\kappa^{ab} n\delta_{m+n,0} + (c_2 + c_4 - g) f^{ab}{}_c (j^c_{z,n+m} + j^c_{\bar{z},n+m}), \tag{5.7}$$

which is a Kac-Moody algebra at level

$$k^+ = -\frac{c_1 - c_3}{(c_2 + c_4 - g)^2} \tag{5.8}$$

as becomes manifest in terms of the rescaled currents

$$\mathcal{J}^a = -i\frac{j^a_z - j^a_{\bar{z}}}{c_2 + c_4 - g}. \tag{5.9}$$

We observe that for the case of the supergroup considered in the earlier section, substituting the values of the c_i in (3.17), we obtain a Kac-Moody algebra at level $k^+ = k$, with the currents taking the simple form

$$\mathcal{J}^a = (j^a_{\bar{z}} - j^a_z). \tag{5.10}$$

When we choose a real form of the supergroup that has a compact subgroup, the level k will be integer. We also observe that the current associated to the σ-component of the canonical right-invariant one-form dgg^{-1} is:

$$\mathcal{J}'^a = c_- j_z + c_+ j_{\bar{z}}. \tag{5.11}$$

In term of these currents, we find the mode algebra:

$$\begin{aligned}
[\mathcal{J}^a_n, \mathcal{J}^b_m] &= -\frac{c_1 - c_3}{(c_2 + c_4 - g)^2}\kappa^{ab} n\delta_{m+n,0} + i f^{ab}{}_c \mathcal{J}^c_{n+m} \\
[\mathcal{J}'^a_n, \mathcal{J}^b_m] &= -i\frac{c_1 c_- + c_3 c_+}{c_2 + c_4 - g}\kappa^{ab} n\delta_{m+n,0} - i f^{ab}{}_c \mathcal{J}'^c_{n+m} \\
[\mathcal{J}'^a_n, \mathcal{J}'^b_m] &= (c_-^2 c_1 - c_+^2 c_3)\kappa^{ab} n\delta_{m+n,0} + f^{ab}{}_c (2c_2 c_- - 2c_4 c_+ + g(c_+ - c_-))\mathcal{J}'^c_{n+m} \\
&\quad - i f^{ab}{}_c (c_2 + c_4 - g)(c_-^2 c_2 + c_4 c_+^2 - g(c_+^2 + c_+ c_- + c_-^2))\mathcal{J}^c_{n+m}.
\end{aligned} \tag{5.12}$$

For the specific case of the supergroup model, we find

$$\begin{aligned}
[\mathcal{J}^a_n, \mathcal{J}^b_m] &= k\kappa^{ab} n \delta_{m+n,0} + i f^{ab}{}_c \mathcal{J}^c_{n+m} \\
[\mathcal{J}'^a_n, \mathcal{J}^b_m] &= \frac{(kf^2-1)(kf^2+1)}{4f^4} \kappa^{ab} n \delta_{m+n,0} - i f^{ab}{}_c \mathcal{J}'^c_{n+m} \\
[\mathcal{J}'^a_n, \mathcal{J}'^b_m] &= 0 \,.
\end{aligned} \tag{5.13}$$

We identified a Kac-Moody subalgebra $\mathcal{J}$ and an infinite set of modes $\mathcal{J}'$ that commute amongst themselves. The latter modes transform into the identity and themselves under the Kac-Moody algebra.

We also note that we can obtain a second Virasoro algebra by applying the Sugawara construction to the Kac-Moody algebra $\mathcal{J}$. The corresponding energy-momentum tensor generates a Virasoro algebra at central charge *sdim* G. It is not holomorphic. The difference of these energy momentum tensors for the left and right group is proportional to the difference of the holomorphic and anti-holomorphic energy momentum tensors. That indicates the existence of a non-chiral analogue of the Knizhnik-Zamolodchikov equation.

6 Conclusions

In this paper, we have performed a generic analysis of the conditions imposed on local Lorentz covariant and PT invariant current algebras. In particular we allowed for parity-breaking models and found a class of solutions to the conditions.

In the case for which the algebra has vanishing Killing form, we showed that one can construct an energy momentum tensor in terms of a current component in a way similar to the Sugawara construction. The current component is then a conformal primary and the central charge is the (super)dimension of the group. This gives a constructive proof of conformality of the quantum model.

We moreover computed exact two- and three-point functions for principal chiral models with Wess-Zumino term for supergroups with vanishing Killing form. Using these exact results, we showed that the current algebra is realized in these models, and we calculated the coefficients in the current algebra. We performed a check on a logarithmic regular term by computing the relevant part of a four-point function. The algebra was independently derived by using the techniques of conformal perturbation theory about the Wess-Zumino-Witten point. We hope the existence of such current algebras will prove useful in furthering the solution of these models [9–11]. Another avenue to explore is to systematically analyze the exactness of low-order perturbation theory for various current and group valued correlators.

One of the examples to which our discussion applies is the sigma model on the supergroup $PSU(1,1|2)$. This particular supergroup is useful to quantize string theory on $AdS_3 \times S^3$ [6, 12]. To quantize the string in the presence of Ramond-Ramond fluxes, we can, in this instance, use the six-dimensional hybrid formalism with eight [28] or sixteen [25] manifest supercharges. In the first case, the $PSU(1,1|2)$ sigma-model is at the core of the worldsheet theory [12].

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It is possible to realize the $AdS_3 \times S^3$ spacetime as the near-horizon limit of a D5-NS5-D1-F1 system. We can then write the parameters of the non-chiral current algebra in terms of the numbers of D5 and NS5 branes [12]. The integer parameter k that multiplies the Wess-Zumino term in the action is equal to the number N_{NS5} of NS5 branes while the parameter $1/f$ is the radius of curvature of spacetime. When the number N_{D5} of D5-branes is equal to zero, the parameters satisfy $kf^2 = 1$ and the non-holomorphic component of the right-invariant current vanishes: we have a chiral current algebra. When we turn on the RR fluxes, we obtain the generic current algebra given in equation (3.26). It is important to further investigate this algebra in the context of string theory on AdS_3. Exploring the integrability of these supergroup models will prove useful in understanding better the properties of the AdS_3/CFT_2 correspondence. The presence of a Kac-Moody algebra at level k over the whole moduli space of the theory may also help in the construction of the string spectrum in $AdS_3 \times S^3$ with Ramond-Ramond fluxes.

Likewise, another application of our analysis is to coset models G/H where G is a supergroup with zero Killing form. In [7] it was shown that a number of coset models where H is a maximal regular subalgebra are conformal to two loops. Graded supercosets based on supergroups with vanishing Killing form are also believed to be conformal [8]. These cosets occur in the worldsheet description of certain string theory backgrounds, for instance, they appear as the central building block of the $AdS_5 \times S^5$ background. Moreover, as symmetric spaces or right coset manifolds, they retain a left group action as a symmetry and we therefore expect that parts of our analysis still apply. It is certainly worth exploring the quantum integrability of these coset models per se, and how it ties in with the conformal current algebra that we have exhibited.

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A Perturbed operator product expansions

We consider the corrections to an OPE induced by an exactly marginal deformation of a conformal field theory. The deformation parameter is denoted by λ. In the deformed theory, we can write the OPE between two operators A and B as:

$$\lim_{z \to w} A(z)B(w) = C(z, w) = \sum_{n \geq 0} \lambda^n C_n(z - w, w)\,, \tag{A.1}$$

where it is implicit that the dependence on the variables does not have to be holomorphic. We expand the result in a basis of operators evaluated at the point w. The operator $C_n(z - w, w)$ is usually written as a series in powers of $z - w$. It is not obvious that the operators A and B (and therefore C) are well-defined operators in the perturbed conformal field theory, but we will assume that that is the case for the model at hand. Let us see how to compute the operators $C_n(z-w,w)$ at order n. By definition, we have:

$$\lim_{z\to w}\langle A(z)B(w)\phi_1(x_1)\dots\phi_p(x_p)\rangle_\lambda=\left\langle\left(\sum_{n\geq 0}\lambda^n C_n(z-w,w)\right)\phi_1(y_1)\dots\phi_p(y_p)\right\rangle_\lambda \tag{A.2}$$

for any operators $\phi_1(y_1)\dots\phi_p(y_p)$. If we want to perform the computation at the non-perturbed point, we write the previous equality as:

$$\begin{aligned}&\lim_{z\to w}\left\langle A(z)B(w)\phi_1(y_1)\dots\phi_p(y_p)\left(\sum_{m\geq 0}\frac{\lambda^m}{m!}\prod_{i=1}^m\int d^2x_i\Phi(x_i)\right)\right\rangle_0\\&=\left\langle\left(\sum_{n\geq 0}\lambda^n C_n(z-w,w)\right)\phi_1(y_1)\dots\phi_p(y_p)\left(\sum_{m\geq 0}\frac{\lambda^m}{m!}\prod_{i=1}^m\int d^2x_i\Phi(x_i)\right)\right\rangle_0\end{aligned} \tag{A.3}$$

where Φ is the exactly marginal operator we use to deform the theory. We isolate the term proportional to λ^n:

$$\begin{aligned}&\lim_{z\to w}\langle A(z)B(w)\phi_1(y_1)\dots\phi_p(y_p)\frac{1}{n!}\prod_{i=1}^n\int d^2x_i\Phi(x_i)\rangle_0\\&=\left\langle\left(\sum_{l=0}^n C_l(z-w,w)\frac{1}{(n-l)!}\prod_{i=1}^{n-l}\int d^2x_i\Phi(x_i)\right)\phi_1(y_1)\dots\phi_p(y_p)\right\rangle_0\end{aligned} \tag{A.4}$$

This becomes an operator identity in the non-perturbed theory:

$$\lim_{z\to w}A(z)B(w)\frac{1}{n!}\prod_{i=1}^n\int d^2x_i\Phi(x_i)=\sum_{l=0}^n C_l(z-w,w)\frac{1}{(n-l)!}\prod_{i=1}^{n-l}\int d^2x_i\Phi(x_i)\,. \tag{A.5}$$

The previous equation defines iteratively the operator C_n which appears in the operator product expansion at order n.

We would like to give a prescription to compute the n-th order term in the OPE, $C_n(z-w,w)$. At zeroth order, the definition is

$$\lim_{z\to w}A(z)B(w)=C_0(z-w,w)\,. \tag{A.6}$$

As expected the zeroth-order OPE is the OPE in the non-deformed model. At order one, we have

$$\lim_{z\to w}A(z)B(w)\int d^2x\Phi(x)=C_0(z-w,w)\int d^2x\Phi(x)+C_1(z-w,w) \tag{A.7}$$

Here is one proposal on how to deal with the left-hand side of this equation. First we let the operator $A(z)$ approach $B(w)$ and $\Phi(x)$ (separately):

$$\lim_{z\to w}A(z)B(w)\int d^2x\Phi(x)=\lim_{z\to w}\left((AB)(z-w,w)\int d^2x\Phi+B(w)\int d^2x(A\Phi)(z-x,x)\right) \tag{A.8}$$

where $(AB)(z-w,w)$ denotes the contraction of $A(z)$ and $B(w)$ (in the unperturbed theory), with the resulting operators evaluated at the point w. It is clear that the first term on the right-hand side is equal to $C_0(z-w,w)\int d^2x\Phi(x)$, so the OPE at first-order is given by the second term:

$$C_1(z-w,w) = \lim_{z\to w} B(w)\int d^2x(A\Phi)(z-x,x)\,. \tag{A.9}$$

At higher order, the same structure appears. We can always recognize in the computation the lower-order contributions, and isolate the highest-order term. We use the definition

$$\lim_{z\to w} A(z)B(w)\frac{1}{n!}\prod_{i=1}^{n}\int d^2x_i\Phi(x_i) = \sum_{l=0}^{n} C_l(z-w,w)\frac{1}{(n-l)!}\prod_{i=1}^{n-l}\int d^2x_i\Phi(x_i)\,. \tag{A.10}$$

To evaluate the left-hand side, we let the operator $A(z)$ approach the other ones. As it approaches $B(w)$, we generate the term with $l=0$ on the right-hand side. As it approaches one of the copies of the marginal operator Φ, we get

$$\lim_{z\to w} B(w)\frac{1}{n!}\int d^2x(A\Phi)(z-x,x)\prod_{i=1}^{n-1}\int d^2x_i\Phi(x_i) \tag{A.11}$$

To carry on, we take the operators $(A\Phi)(z-x,x)$ that was just generated at the point x and let it approach the other operators in the expression. If it approaches $B(w)$, then we generate the term with $l=1$ in the right-hand side of the definition. Otherwise we generate a new expression on which we apply the same procedure.

Finally, we understand how to obtain directly the order-n OPE $C_n(z-w)$: it is the term that we get by first contracting $A(z)$ with all the integrated operators, and then contracting with $B(w)$ at the very end. We will denote it as:

$$C_n(z-w,w) = \left[A(z)\frac{1}{n!}\prod_{i=1}^{n}\int d^2x_i\Phi(x_i)\right]B(w) \tag{A.12}$$

All the operators inside the square brackets have to be contracted, before performing the last contraction with the operator outside the square brackets.

We should stress that the previous procedure is not always well-defined. In the computation described in the bulk of this paper, this prescription leads to an unambiguous result for the poles of the current-current OPEs. However, in a more general context, the integrals appearing in the above calculations need a more careful regularization.

B Detailed operator product expansions

In this appendix we show how to compute OPEs involving the operator $(g\bar{J}g^{-1})$ in the WZW model.

The contact terms. It is natural to postulate contact terms between the left- and right-invariant currents (see e.g. [29]). Indeed, even for a U(1) current algebra, contact terms can be derived from the representation of the current algebra in terms of a free boson and its logarithmic propagator. Since at large level k, the group manifold flattens and is equivalent to a set of free fields, we do expect contact terms to arise. We propose the following contact terms:

$$J^a(z,\overline{z})(g\overline{J}g^{-1})^b(w,\overline{w}) \sim 2\pi k\kappa^{ab}\delta^{(2)}(z-w) + \cdots \tag{B.1}$$

The Maurer-Cartan equation in the quantum theory. In the quantum theory, the composite operator in the Maurer-Cartan equation is ambiguous due to normal ordering. With our choice of normal ordering, it is natural to propose the quantum Maurer-Cartan equation

$$\overline{\partial}J^c + \partial(g\overline{J}g^{-1})^c + \frac{i}{2k}f^c{}_{de}\left(: J^d(g\overline{J}g^{-1})^e : + : \left(g\overline{J}g^{-1}\right)^e J^d :\right) = 0\,. \tag{B.2}$$

One way to check this proposal is to compute the OPE between the current components J^a and the operator on the left hand side of equation (B.2) which is classically zero due to the Maurer-Cartan equation. In the calculation, it is crucial to apply the normal ordering prescription we introduced in section 2. We not only confirm the above proposal for the quantum Maurer-Cartan equation, but also find that we need to fix the OPE between J^a and $(g\overline{J}g^{-1})^b$ to be

$$J^a(z)(g\overline{J}g^{-1})^b(w,\overline{w}) \sim 2\pi\kappa^{ab}\delta^{(2)}(z-w) + if^{ab}{}_c\frac{(g\overline{J}g^{-1})^c(w,\overline{w})}{z-w} + : J^a(g\overline{J}g^{-1})^b : (w,\overline{w})\,. \tag{B.3}$$

Let us show this calculation in some detail in order to illustrate the techniques involved. Using the holomorphy of the current J and the knowledge of the naive conformal dimensions of the operators, we can make the ansatz

$$J^a(z)(g\overline{J}g^{-1})^b(w,\overline{w}) = 2\pi k\kappa^{ab}\delta^{(2)}(z-w) + \alpha i f^{abc}\frac{(g\overline{J}g^{-1})^c(w,\overline{w})}{z-w} + : J^a(g\overline{J}g^{-1})^b : (w,\overline{w}) \tag{B.4}$$

With the definition of the normal ordering above, let us compute the operator product expansion between $J^a(z)$ and the Maurer-Cartan equation. We distinguish two terms. The first term is

$$\begin{aligned}
J^a(z)\left(\overline{\partial}J^c(w) + \partial(g\overline{J}g^{-1})^c(w,\overline{w})\right) = &\\
\overline{\partial}_w\left(\frac{k\kappa^{ac}}{(z-w)^2} + if^{acd}\frac{J^d(w)}{z-w}\right)&\\
+\partial_w\left(2\pi k\kappa^{ac}\delta^{(2)}(z-w) + \alpha i f^{acd}\frac{(g\overline{J}g^{-1})^d(w,\overline{w})}{z-w}\right) + \cdots&\\
= -if^{acd}J^d(w)2\pi\delta^{(2)}(z-w) + \alpha i f^{acd}\frac{(g\overline{J}g^{-1})^d(w,\overline{w})}{(z-w)^2}&\\
+if^{acd}\frac{(\alpha\,\partial(g\overline{J}g^{-1})^d(w,\overline{w}) + \overline{\partial}J^d(w))}{z-w} + \cdots&
\end{aligned} \tag{B.5}$$

From the last terms we see that we can obtain a pole term proportional to the Maurer-Cartan equation if we put $\alpha = 1$. It can be shown that this is the only consistent possibility, and we will freely put $\alpha = 1$ from now on. The second term with an extra minus sign is given by

$$-\frac{if^c_{de}}{k}J^a(z)\lim_{:x\to w:}(J^d(x)(g\overline{J}g^{-1})^e(w,\overline{w}) + (g\overline{J}g^{-1})^e(w,\overline{w})J^d(x)) \,. \tag{B.6}$$

where we have used the normal ordering prescription. Let us start with the first of the two terms in (B.6) (suppressing the overall $-\frac{i}{k}f^c_{de}$):

$$\begin{aligned} J^a(z)\lim_{:x\to w:}J^d(x)(g\overline{J}g^{-1})^e(w,\overline{w}) = \lim_{:x\to w:}\Bigg\{&\left[\frac{k\kappa^{ad}}{(z-x)^2} + i\frac{f^{ad}_g J^g(x)}{(z-x)}\right](g\overline{J}g^{-1})^e(w,\overline{w}) \\ &+J^d(x)\left[2\pi k\kappa^{ae}\delta^{(2)}(z-w) + i\alpha\frac{f^{ae}_g(g\overline{J}g^{-1})^g(w,\overline{w})}{(z-w)}\right]\Bigg\} \,. \end{aligned} \tag{B.7}$$

We perform successive contractions, and subtract singular terms according to the normal ordering procedure to obtain

$$\begin{aligned} &-2i\pi f^{ca}_d\delta^{(2)}(z-w)J^d(w) - \frac{i(f^{ca}_d\delta^d_h - \frac{\alpha}{k}f^c_{de}f^{ad}_g f^{ge}_h)(g\overline{J}g^{-1})^h(w,\overline{w})}{(z-w)^2} \\ &\qquad + \frac{f^c_{de}(f^{ad}_g : J^g(g\overline{J}g^{-1})^e : (w,\overline{w}) + \alpha f^{ae}_g : J^d(g\overline{J}g^{-1})^g : (w,\overline{w}))}{k(z-w)} \,. \end{aligned} \tag{B.8}$$

When $\alpha = 1$ and using the Jacobi identity, we can simplify further:

$$-2\pi i f^{ca}_d\delta^{(2)}(z-w)J^d(w) - i\left(1+\frac{2\hat{h}}{k}\right)\frac{f^{ca}_d(g\overline{J}g^{-1})^d}{(z-w)^2} - \frac{f^{ca}_d f^d_{ge} : J^g(g\overline{J}g^{-1})^e : (w,\overline{w})}{k(z-w)} \,. \tag{B.9}$$

Analogously, the second part of the second term becomes

$$\begin{aligned} &-2\pi i f^{ca}_d\delta^2(z-w)J^d(w) - i\left(1-\frac{2\hat{h}}{k}\right)\frac{(f^{ca}_d(g\overline{J}g^{-1})^d}{(z-w)^2} \\ &\qquad + \frac{f^c_{de}}{k(z-w)}\Big(f^{ae}_g : (g\overline{J}g^{-1})^g J^d : (w,\overline{w}) + f^{ad}_g : (g\overline{J}g^{-1})^e J^g : (w,\overline{w})\Big) \,. \end{aligned} \tag{B.10}$$

Combining the two parts of the second term we find

$$\begin{aligned} &2\pi i f^{ac}_d\delta^2(z-w)J^d(w) + \frac{if^{ac}_d(g\overline{J}g^{-1})^d}{(z-w)^2} \\ &\qquad + \frac{f^{ac}_d f^d_{eg}(: (g\overline{J}g^{-1})^g J^e : (w,\overline{w}) + : J^e(g\overline{J}g^{-1})^g : (w,\overline{w}))}{2k(z-w)} \,. \end{aligned} \tag{B.11}$$

Comparing with the first term, we see that the contact term as well as the double pole term cancel exactly while the single pole term vanishes using the Maurer-Cartan equation itself, normal ordered as in our proposal. We note that the operator product expansion between

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J^a and $(g\overline{J}g^{-1})^b$ obtained this way matches the operator product expansion obtained in (3.15) at the Wess-Zumino-Wien point.

We will also need the OPE of $(g\overline{J}g^{-1})^a(z,\overline{z})$ with itself at the Wess-Zumino-Witten point:

$$\left(g\overline{J}g^{-1}\right)^a(z,\overline{z})(g\overline{J}g^{-1})^b(w,\overline{w}) \sim \frac{k\kappa^{ab}}{(\overline{z}-\overline{w})^2} + \frac{if_c^{ab}(z-w)J^c(w)}{(\overline{z}-\overline{w})^2} - 2\frac{if_c^{ab}(g\overline{J}g^{-1})^c(w,\overline{w})}{\overline{z}-\overline{w}}\,. \tag{B.12}$$

The coefficients can be argued for by analyzing the contact and most singular terms in the OPE of the current $g\overline{J}g^{-1}$ with the Maurer-Cartan equation.

Operator product expansions of currents with marginal operator. Since the computations are fairly similar, let us consider the more complicated OPE of the current $(g\overline{J}g^{-1})^a(w,\overline{w})$ with the marginal operator Φ. The first part of the computation involves the OPE

$$\begin{aligned}
&\left(g\overline{J}g^{-1}\right)^b(w,\overline{w}) \lim_{:y\to x:} J^c(y)(g\overline{J}g^{-1})_c(x,\overline{x}) \sim \\
&\quad \lim_{:y\to x} \Bigg\{ \left(2\pi k\delta^{(2)}(w-y) + \frac{if_d^{bc}(g\overline{J}g^{-1})^d(y,\overline{y})}{w-y}\right)(g\overline{J}g^{-1})_c(x,\overline{x}) \\
&\quad +J^c(y)\left(\frac{k\delta_c^b}{(\overline{w}-\overline{x})^2} + \frac{if_{cd}^b(w-x)J^d(x)}{(\overline{w}-\overline{x})^2} - \frac{2if_{cd}^b(g\overline{J}g^{-1})^d(x,\overline{x})}{\overline{w}-\overline{x}}\right)\Bigg\} \\
&\sim 2\pi k\delta^{(2)}(w-x)(g\overline{J}g^{-1})^b(x,\overline{x}) + \frac{kJ^b(x)}{(\overline{w}-\overline{x})^2} - \frac{2if_{cd}^b}{(\overline{w}-\overline{x})} : J^c(g\overline{J}g^{-1})^d : (x,\overline{x}) \\
&\quad + \frac{if_{cd}^b(w-x)}{(\overline{w}-\overline{x})^2} : J^cJ^d : (x) - \frac{if_{cd}^b}{w-x} : (g\overline{J}g^{-1})^c(g\overline{J}g^{-1})^d : (x,\overline{x})\,.
\end{aligned} \tag{B.13}$$

Similarly, exchanging the order of J and $g\overline{J}g^{-1}$, we find an identical OPE to the above one except that the terms in the last line have opposite sign. Combining these two, we therefore find that

$$\begin{aligned}
\left(g\overline{J}g^{-1}\right)^b(w,\overline{w})\Phi(x,\overline{x}) &\sim 2\pi k\delta^{(2)}(w-x)\left(g\overline{J}g^{-1}\right)^b(x,\overline{x}) + \frac{kJ^b(x)}{(\overline{w}-\overline{x})^2} \\
&\quad -\frac{2if_{cd}^b}{\overline{w}-\overline{x}}\left(:\left(g\overline{J}g^{-1}\right)^d J^c : + : J^c\left(g\overline{J}g^{-1}\right)^d :\right)(x,\overline{x})\,.
\end{aligned} \tag{B.14}$$

Now, the last term can be rewritten using the Maurer-Cartan identity and we obtain

$$\begin{aligned}
\left(g\overline{J}g^{-1}\right)^b(w,\overline{w})\Phi(x,\overline{x}) &\sim 2\pi k\delta^{(2)}(w-x)\left(g\overline{J}g^{-1}\right)^b(x,\overline{x}) + \frac{kJ^b(x)}{(\overline{w}-\overline{x})^2} \\
&\quad +\frac{2k}{\overline{w}-\overline{x}}\left(\overline{\partial}J^b + \partial(g\overline{J}g^{-1})^b\right)(x,\overline{x})\,.
\end{aligned} \tag{B.15}$$

Integrating over the location of the marginal operator and using the identities

$$\int d^2x\frac{\overline{\partial}J^b}{\overline{w}-\overline{x}} = -\int d^2x\frac{J^b(x)}{(\overline{w}-\overline{x})^2} \tag{B.16}$$

$$\int d^2x\frac{\partial(g\overline{J}g^{-1})^b}{\overline{w}-\overline{x}} = 2\pi(g\overline{J}g^{-1})^a(w,\overline{w})\,, \tag{B.17}$$

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we find the contraction

$$\left(g\overline{J}g^{-1}\right)^b(w,\overline{w})\int d^2x\Phi(x,\overline{x}) = 6\pi k\left(g\overline{J}g^{-1}\right)^b(w,\overline{w}) - k\int d^2x\frac{J^b(x)}{(\overline{w}-\overline{x})^2}\,. \tag{B.18}$$

For the OPE of $J^a(w)$ with the marginal operator, it turns out that both orderings lead to the same answer, so we only exhibit the following OPE:

$$\begin{aligned}
J^b(w)\lim_{:y\to x:}:J^c(y)(g\overline{J}g^{-1})_c(x,\overline{x}):\sim &\lim_{:y\to x:}\left\{\left(\frac{k\kappa^{bc}}{(w-y)^2}+\frac{if^{bc}_d J^d(y)}{w-y}\right)(g\overline{J}g^{-1})_c(x,\overline{x})\right.\\
&\left.+J^c(y)\left(2\pi k\delta^{(2)}(w-x)\delta^b_c+\frac{if^b_{cd}(g\overline{J}g^{-1})^d(x,\overline{x})}{w-y}\right)\right\}\\
\sim&\frac{k(g\overline{J}g^{-1})^b(x,\overline{x})}{(w-y)^2}+2\pi k\delta^{(2)}(w-x)J^b(x)\\
&+\frac{if^b_{cd}}{w-y}(:J^d(g\overline{J}g^{-1})^c:+:J^c(g\overline{J}g^{-1})^d:)(x,\overline{x})\\
\sim&2\pi k\delta^{(2)}(w-x)J^b(x)+\frac{k(g\overline{J}g^{-1})^b(x,\overline{x})}{(w-y)^2}.
\end{aligned} \tag{B.19}$$

C Useful integrals

We tabulate a few useful integrals that have been used throughout the article (see e.g. [29]):

$$\int\frac{d^2x}{(\overline{x}-\overline{w})(x-z)} = -2\pi\log|z-w|^2 \tag{C.1}$$

$$\int\frac{d^2x}{(\overline{x}-\overline{w})^2(x-z)} = 2\pi\frac{1}{\overline{z}-\overline{w}} \tag{C.2}$$

$$\int\frac{d^2x}{(\overline{x}-\overline{w})(x-z)^2} = -2\pi\frac{1}{z-w} \tag{C.3}$$

$$\int\frac{d^2x}{(\overline{x}-\overline{w})^2(x-z)^2} = 4\pi^2\delta^{(2)}(z-w) \tag{C.4}$$

$$\int\frac{d^2x}{(z-x)^2(w-x)} = -2\pi\frac{\overline{z}-\overline{w}}{(z-w)^2}\,. \tag{C.5}$$

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